N. Jacobsen LGS Innovations

The following is a description the Maximum Likelihood (ML) decision rule for detecting whether or not a coded signal is present in a noisy received signal. The signal $\mathbf{x}$ is assumed to be from the code $\mathcal{C}$, consisting of a length N BPSK codeword. When a codeword is sent by the transmitter, the receiver signal is modeled as

$$\mathbf{y} = a\mathbf{x} + \mathbf{w}, \tag{1}$$

where a denotes a channel fading coefficient and $\mathbf{w}$ is a vector of i.i.d. complex Gaussian random variables with zero-mean and variance $2\sigma^2$. The conditional probability density of $\mathbf{y}$ given that $\mathbf{x}$ is the transmitted codeword and a is the channel scale factor is

$$f(\mathbf{y}|\mathbf{x}, a) = \prod_{n=1}^{N} \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}|y_n - a x_n|^2}, \tag{2}$$

where x_n denotes the nth element of the vector $\mathbf{x}$.

Assuming a has the probability density function $f(a)$, the conditional likelihood of observing $\mathbf{y}$ given $\mathbf{x}$ is:

$$f(\mathbf{y}|\mathbf{x}) = \int_{\mathcal{A}} f(\mathbf{y}|\mathbf{x}, a) f(a) da,$$

where $\mathcal{A}$ denotes the support of a.

When signal is absent, the receiver observes only noise:

$$\mathbf{y} = \mathbf{w}, \tag{3}$$

The basic question addressed is whether a coded signal is present, i.e. the case of (1) for some $\mathbf{x} \in \mathcal{C}$, or whether no signal is present, i.e. the case of (3). Let H_1 denote the hypothesis that equation (1) is true and let H_0 denote the hypothesis that equation (3) is true (the "null hypothesis"). The ML decision rule is defined as:

$$\hat{b} = \arg \max_{b \in \{0,1\}} f(\mathbf{y}|H_b),$$

where $\hat{b}$ denotes the index of the most likely hypothesis and $f(\mathbf{y}|H_b)$ denotes the conditional likelihood of observing $\mathbf{y}$ given that hypothesis H_b is true.

Hence we need to find $f(\mathbf{y}|H_b)$ for $b = 0,1$. For the $b = 0$ case, $f(\mathbf{y}|H_0) = f(\mathbf{y}|\mathbf{x}, a = 0)$, where $f(\mathbf{y}|\mathbf{x}, a)$ is defined in (2). For $b = 1$:

$$\begin{aligned} f(\mathbf{y}|H_1) &= \sum_{\mathbf{x} \in \mathcal{C}} f(\mathbf{y}|\mathbf{x}) P(\mathbf{x}) \\ &= \frac{1}{|\mathcal{C}|} \sum_{\mathbf{x} \in \mathcal{C}} \int_{\mathcal{A}} f(\mathbf{y}|\mathbf{x}, a) f(a) da \qquad (4) \\ &= E\left(\frac{1}{|\mathcal{C}|} \sum_{\mathbf{x} \in \mathcal{C}} f(\mathbf{y}|\mathbf{x}, a) \right). \qquad (5) \end{aligned}$$

In equation (5), $E(*)$ denotes the expected value with respect to the probability density of a. Hence, $f(\mathbf{y}|H_1)$ can be computed by Monte-Carlo integration with respect to the pdf of a, with the following integrand:

$$\begin{aligned} f(\mathbf{y}|H_1,a) &= \frac{1}{|\mathcal{C}|}\sum_{\mathbf{x}\in\mathcal{C}} f(\mathbf{y}|\mathbf{x},a) \\ &= \frac{1}{|\mathcal{C}|}\left(\frac{1}{2\pi\sigma^2}\right)^N \sum_{\mathbf{x}\in\mathcal{C}}\prod_{n=1}^{N} e^{-\frac{1}{2\sigma^2}|y_n-ax_n|^2} \\ &= \frac{1}{|\mathcal{C}|}\left(\frac{1}{2\pi\sigma^2}\right)^N e^{-\frac{1}{2\sigma^2}(\|\mathbf{y}\|^2+a^2N)}\sum_{\mathbf{x}\in\mathcal{C}} e^{\frac{a}{\sigma^2}\mathrm{Re}(\mathbf{x}^{\mathrm{H}}\mathbf{y})}, \end{aligned} \tag{6}$$

where $(*)^{\mathrm{H}}$ denotes the conjugate transpose.

Hence, the receiver decides whether the signal is present or not based on the maximum of $f(\mathbf{y}|H_0)$ and $f(\mathbf{y}|H_1)$. Note that in equation (4), equiprobable codewords are assumed. However, equation (5) can be modified to include a non-uniform distribution over the codeword set ("prior information"). Furthermore, (5) can be calculated based on an estimate of the channel coefficient rather than its pdf.

Noting that $f(\mathbf{y}|H_0) = \left(\frac{1}{2\pi\sigma^2}\right)^N e^{-\frac{1}{2\sigma^2}\|\mathbf{y}\|^2}$ appears as a factor in (5), an equivalent decision statistic can be formulated as:

$$\lambda(\mathbf{y}) \triangleq \frac{f(\mathbf{y}|H_1)}{f(\mathbf{y}|H_0)} = E\left(\frac{1}{|\mathcal{C}|}e^{-\frac{a^2N}{2\sigma^2}}\sum_{\mathbf{x}\in\mathcal{C}} e^{\frac{a}{\sigma^2}\mathrm{Re}(\mathbf{x}^{\mathrm{H}}\mathbf{y})}\right).$$

Accordingly the receiver chooses hypothesis one if $\lambda(\mathbf{y}) > 1$ and hypothesis zero otherwise.