Practical Cooperative Coding for Half-Duplex Relay Channels

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***Abstract*—Simple variations on rate-compatible channel codes are shown to achieve cooperative coding gains for the half-duplex relay channel without the complexity of capacity approaching codes. The simulated performance of an optimized irregular low density parity check code is provided.**

I. Introduction

The relay channel has been studied actively by the information theory community since pioneering work by Cover and El Gamal in the 1970s [1]. The capacity of the relay channel, though characterized for certain specific cases, is still an open problem. Nonetheless, relays are commonly deployed in wireless radio systems for range extension and coverage enhancement. (For example, wireless repeaters are commonly used in amateur radio systems.)

Several cooperative communication strategies that allow for cooperation between the source and relay transmitters are known to approach capacity depending on the relay geometry and system assumptions [2]. Here the focus is on the decode-and-forward relay strategy, where the relay must reliably decode the source message. Note that an alternative strategy, known as the compress-and-forward strategy, which requires a reliable relay-destination link, is known to be preferable in terms of capacity when the relay is relatively close to the destination [2].

In a decode-and-forward system, the relay is required to decode all or part of the source message. Using knowledge of the decoded message bits, the relay can transmit coded bits that cooperatively enable higher communication rates and/or desirable power emission behavior as compared to a point-to-point (non-relay enabled) system. The decode-and-forward strategy yields capacity approaching performance for many important system geometries and is known to outperform the canonical amplify-and-forward relay strategy in all known examples.

The relay is assumed to be half-duplex: it does not simultaneously transmit and receive radio signals. The half-duplex relay model well suits current radio hardware limitations. In this work it is shown that practical cooperative codes, based on rate-compatible error correcting codes, are able to perform well with respect to the information-theoretic benchmarks for the half-duplex relay channel. A new family of rate-compatible irregular Low Density Parity Check (LDPC) codes, termed Time Division Multiple Access (TDMA) relay LDPC codes, is introduced. The performance of the capacity approaching code is shown to be well approximated by the practical TDMA cooperative code. For rates between 0.1 and 0.6, the computer simulated performance shows as much as 5 dB gained over the point-to-point irregular LDPC code (using Binary Phase Shift Keying (BPSK) modulation).

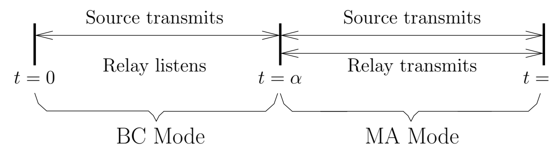

Fig. 1. Time-axis representation for the half-duplex relay model.

II. Channel model and capacity results

Figure 1 depicts a time-axis representation of the half-duplex relay channel. During the first time slot, termed Broadcast (BC) mode, the relay and destination receivers listen to the source transmitted symbol. During the second time slot, termed Multiple-Access (MA) mode, the source and relay transmit simultaneously regarding the source message. The time-sharing parameter between BC and MA modes, α, is chosen to maximize the rate.

All links are assumed to represent complex baseband Additive White Gaussian Noise (AWGN) channels. A total power constraint, P, is imposed, such that $P = P_{BC} + P_{MA}$, where P_{BC} and P_{MA} represent the total system power during BC and MA modes, respectively. In order to evaluate a benefit for using the relay, comparison is made to a point-to-point link with the same total power P. For convenience, a power-sharing parameter, β, is defined, such that $P_{BC} = \beta P$ and $P_{MA} = (1 - \beta)P$.

For simplicity, the channel geometry is modeled as linear, with the relay at distance d from the source, on a unit-line between the source and destination. See Figure 2. Link attenuations are modeled with scalar amplitude gain factors, c_1 (for the source-relay link), c_2 (source-destination), and c_3 (relay-destination), that vary as $d^{-p/2}$, where d denotes the distance and p denotes the Radio Frequency (RF) path loss exponent. Without loss of generality, the relay and destination additive receiver noise, N_R and N_D, respectively, are modeled as zero-mean, independent and identically distributed (i.i.d.),

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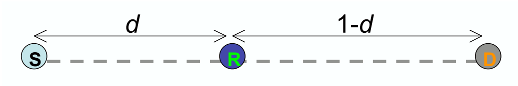


Fig. 2. Linear relay geometry. All links are assumed to have the same path loss exponent.

Gaussian random variables with unit variance.

The relay and destination received symbols are given by:

$$Y_R = c_1 X_1 + N_R, \tag{1}$$
$$Y = c_2 X + c_3 [0\ X_R] + N_D, \tag{2}$$

where the source symbol $X = [X_1\ X_2]$ is defined in terms of its BC component X_1 and its MA component X_2, and the MA mode relay symbol is given by X_R. Then, $P_{BC} = E(|X_1|^2)$ and $P_{MA} = E(|X_2|^2) + E(|X_R|^2)$.

First, the best known capacity bounds for the half-duplex relay channel are discussed. In the general case, during MA mode, the relay transmits cooperatively (in the same frequency band) with the source. In this paper, the relay symbol depends only on the source symbol it has received during the previous BC mode. For the purpose of capacity evaluation, one BC/MA coding cycle in the relay system is compared to a single channel use in a point-to-point system with the same total power and time. In the following expressions, Shannon's Gaussian channel coding formula is defined in terms of the Signal-to-Noise Ratio (SNR), x, as $C(x) = \log(1 + x)$.

A. Capacity upper bound

The upper bound on capacity for the half-duplex model is given by the max-flow min-cut theorem of network information theory [3]. This upper bound is the capacity of a physically degraded relay channel, and for the relay channel with feedback [1].

$$C \leq \max_{p(X,X_R)} \min\{I(X;Y,Y_R|X_R), I(X,X_R;Y)\}. \tag{3}$$

The half-duplex Gaussian case is recalled here. In cut one, which excludes the source node, the capacity is bounded by rate R_1, where

$$R_1 = \alpha C\left(c_1^2 E_{BC}\right) + \alpha C\left(c_2^2 E_{BC}\right) + (1-\alpha) C\left(c_2^2 (1-\gamma^2)(1-\delta) E_{MA}\right), \tag{4}$$

where

$$\delta = \frac{E(|X_R|^2)}{P_{MA}}, \tag{5}$$

represents the MA mode power-share between the source and relay symbols,

$$\gamma = \frac{E(X_2 X_R^*)}{\sqrt{E(|X_2|^2) E(|X_R|^2)}}, \tag{6}$$

denotes the correlation coefficient for the source and relay MA mode symbols, $E_{BC} = P_{BC}/\alpha$ and $E_{MA} = P_{MA}/(1-\alpha)$.

In cut two, which excludes the destination node, the capacity is bounded by rate R_2, where

$$R_2 = \alpha C\left(c_2^2 E_{BC}\right) + (1-\alpha) C\left(\frac{h^T \Sigma h}{1-\alpha}\right). \tag{7}$$

In the above expression, $h = [c_3\ \ c_2]^T$ denotes the MA mode vector channel, and $\Sigma = E(X_{MA} X_{MA}^H)$ denotes the covariance of the MA mode symbols

$$X_{MA} = [X_R\ X_2]^T, \tag{8}$$

where

$$\Sigma = P_{MA} \begin{bmatrix} \delta & \gamma\sqrt{\delta(1-\delta)} \\ \gamma\sqrt{\delta(1-\delta)} & 1-\delta \end{bmatrix}. \tag{9}$$

The max-flow min-cut bound is then given by

$$C \leq \max_{\alpha,\beta,\gamma,\delta} \min\{R_1, R_2\}. \tag{10}$$

Next, it is shown that this upper bound can be approached using a decode-and-forward strategy. Decode-and-forward is shown to out-perform all major known strategies for the important geometric case considered here. The decode-and-forward achievable rate relies on phase-synchronous reception of the source and relay symbols and successive interference cancellation at the receiver.

B. Decode-and-forward achievable rate

In the decode-and-forward achievable rate, see e.g. [4], the source message W is split into two parts: W_r, representing R_r bits, communicated using the relay, and W_d, representing R_d bits, communicated direct to the destination. The total rate is given by the sum of the component rates, $R = R_r + R_d$.

For the relay to decode the relay symbol, W_r, the relay component code rate, R_r, is bounded by the source-relay capacity, C_{r1}, where

$$C_{r1} = \alpha C\left(c_1^2 E_{BC}\right). \tag{11}$$

Next, the destination is assumed to decode W_r treating the received code symbol for W_d as noise. This is possible, assuming $R_r \leq C_{r2}$, where

$$C_{r2} = \alpha C\left(c_2^2 E_{BC}\right) + (1-\alpha) C\left(S_1\right), \tag{12}$$

where

$$S_1 = \frac{\left(c_3\sqrt{\delta E_{MA}} + c_2 \gamma \sqrt{(1-\delta) E_{MA}}\right)^2}{1 + c_2^2 (1-\gamma^2)(1-\delta) E_{MA}}. \tag{13}$$

Note that the above rate (12) assumes that the relay and source symbols are received phase synchronous by the destination receiver.

Finally, the direct symbol W_d can be decoded if $R_d \leq C_d$, where

$$C_d = (1-\alpha) C\left(c_2^2 (1-\gamma^2)(1-\delta) E_{MA}\right). \tag{14}$$

Note that successive interference canceling is used to decode the MA mode received symbols.

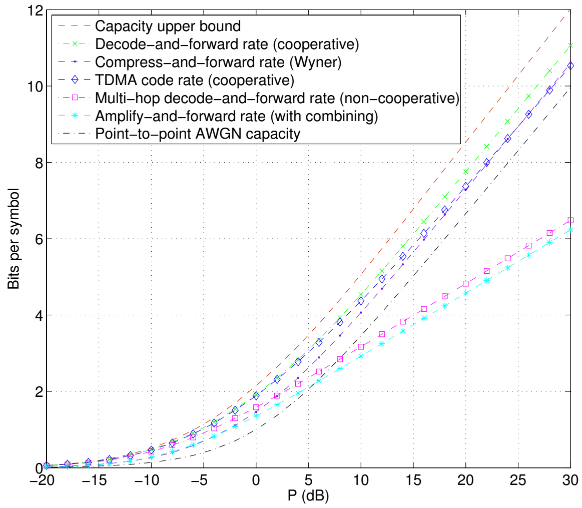


Fig. 3. Known bounds for the half-duplex relay channel.

The decode-and-forward rate R_{DF} is then achievable if

$$R_{DF} \leq \min\{C_{r1}, C_{r2}\} + C_d, \tag{15}$$

and is maximized by optimizing the parameters α, β, γ and δ.

C. TDMA decode-and-forward rate

The TDMA relay strategy is a simplified decode-and-forward strategy in which the source does not transmit during MA mode. The source and relay symbols are received orthogonal, with $P_{MA} = E(|X_R|^2)$.

The overall channel seen by the destination receiver in the TDMA model is comprised as a mixture of AWGN channels (the relay-destination and source-destination channels). The achieved rate is given as an average of the capacities of the two channels, in which the time-sharing parameter, α, and power-sharing parameter, β, are optimized. In decode-and-forward, the source-relay code rate is bounded by the source-relay capacity. The overall rate, R_{TDMA}, is given by

$$\begin{aligned} R_{TDMA} = \max_{\alpha,\beta} \min\{&\alpha C\left(c_2^2 E_{BC}\right) + (1-\alpha) C\left(c_3^2 E_{MA}\right), \\ &\alpha C\left(c_1^2 E_{BC}\right)\}. \end{aligned} \tag{16}$$

D. Multi-hop code rate

Multi-hop is a non-cooperative decode-and-forward strategy in which the destination does not decode the direct received symbol. The multi-hop strategy serves as a benchmark for comparing the performance of different cooperative coding strategies. The channel coding is assumed to be independent on the two links, and the overall rate is given by the minimum of the capacities of the two links:

$$R_{MH} = \max_{\alpha,\beta} \min\{\alpha C\left(c_1^2 E_{BC}\right), (1-\alpha) C\left(c_3^2 E_{MA}\right)\}. \tag{17}$$

E. Amplify-and-forward rate

In the amplify-and-forward strategy, during MA mode the relay simply transmits a scaled version of the BC mode received symbol. The amplify-and-forward strategy requires neither the relay to decode the source message nor a reliable relay-destination link. The receiver performs Maximal Ratio Combining (MRC) with the BC mode and MA mode received symbols. The time-sharing parameter is assumed to be one-half. The capacity is given by

$$R_{AF} = \max_{\beta} \frac{1}{2} C\left((c_2^2 + \frac{c_1^2 A_1}{A_1 + 1}) E_{BC} \right), \tag{18}$$

where $A_1 = \frac{c_3^2 E_{MA}}{c_1^2 E_{BC} + 1}$.

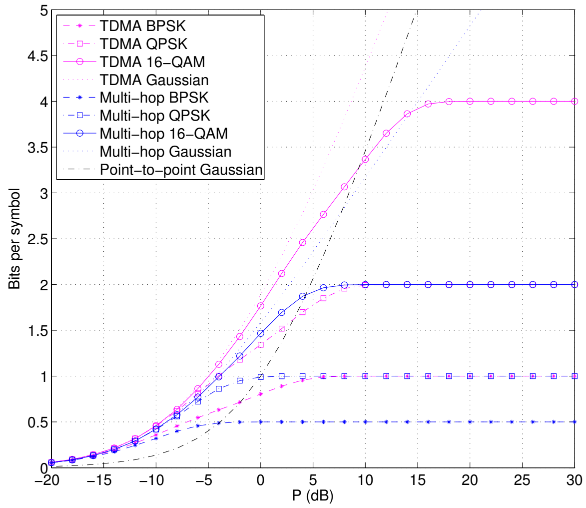
III. Numerical Example of the Capacity Bounds

Figure 3 contains a numerical evaluation of the capacity bounds for the linear geometry with distance $d = 1/2$ and path loss exponent $p = 3$. The decode-and-forward achievable rate is shown to approach the capacity upper bound especially at low system power gain (P). Thus, a well known albeit complex decode-and-forward code is able to approach the capacity within a reasonable gap for the linear geometry considered here. Further, the TDMA code rate closely approximates the decode-and-forward achievable rate over a wide range of SNR. Thus, a cooperative coding benefit is obtained (i.e., rates are achieved beyond the point-to-point curve) using a simple rate-compatible code structure that does not require phase synchronous reception of the source and relay signals or interference canceling decoding (both of which are used in the decode-and-forward achievable rate). Also plotted is the rate achieved by the compress-and-forward strategy, using results from [5] and [6], as detailed in Appendix A.

To illustrate the above claims, the system power gain is compared for the different cooperative strategies at the spectral efficiency of 4 bits per channel symbol. At this rate, from the capacity upper bound, the decode-and-forward achievable rate is at most 1.51 dB from capacity. The gap between the TDMA relay code and decode-and-forward is measured at 0.44 dB, whereas the compress-and-forward achievable rate is measured at 1.56 dB from decode-and-forward. Lastly, at 4 bits per symbol, the TDMA relay code is 3.10 dB better than the point-to-point code.

The TDMA and multi-hop decode-and-forward rates attained using specific input alphabets are compared in Figure 4. The curves show a large gain in spectral efficiency for the TDMA code over multi-hop for a given modulation alphabet. In the multi-hop code, the destination does not decode the BC mode received symbol and thus no cooperative coding benefit can be obtained. The TDMA code rate is better than the point-to-point capacity by as much as 5 dB while doubling the high-SNR QAM rate of multi-hop.

The above comparisons show that the TDMA relay coding strategy yields comparable performance to the best known half-duplex relay strategies for the specific channel model



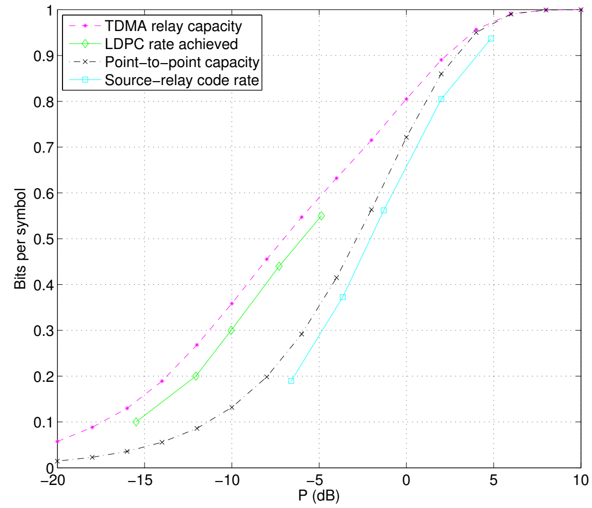


Fig. 4. Rate of TDMA relay code compared to multi-hop for various input alphabets.

Fig. 5. Performance of TDMA relay LDPC code with BPSK inputs.

considered here. Further computations are required to fully characterize the performance for other geometries of interest.[1]

IV. LDPC CODE DESIGN FOR TDMA RELAY

The optimal channel coding strategy for the relay to employ in the TDMA framework is to transmit compatible code bits during MA mode for the source code bits it has decoded in BC mode. Performance data for a specifically designed LDPC code, consisting of jointly optimized rate-compatible parity check matrices, for use with BPSK modulation, is provided. Other modulation alphabets are readily accommodated by the optimization framework (see [7] for details). Note that off-the-shelf rate-compatible error-correcting codes, including standardized H-ARQ turbo-codes, are consistent with the TDMA coding framework but would not exhibit the same performance benefit as the optimized code.

The Edge Growth and Parity Splitting (EG/PS) technique, from [8], is used here to develop rate-compatible LDPC parity matrices for use by the source and relay encoders. The rate of the BC mode parity matrix, used by the source encoder, is bounded by the source-relay link capacity. The relay, having decoded the BC mode code word, generates the MA mode code word based on a larger, rate-compatible parity matrix, whose size corresponds to the overall communication rate achieved by the code. The optimization algorithm jointly maximizes these two rates for the given geometry and total system power constraint. The effective channel observed by the destination receiver is characterized by a mixture of AWGN channels, parameterized by the source-destination and relay-destination SNR. In optimizing the parity matrices, the base parity matrix employed by the source encoder imposes a constraint on the parity check matrix employed by the relay encoder. The relay encoder degree distribution optimization is based on Extrinsic Information Transfer (EXIT) chart techniques [9] (see Appendix B).

A. Computer simulation results

Figure 5 provides the performance of the TDMA relay LDPC code using BPSK modulation. A linear relay geometry is assumed, with distance $d = 1/2$ and path loss exponent $p = 3$. The time- and power- sharing parameters correspond to their optimal values as given by the rate maximization (16). In the achieved rate, two link capacities are approached simultaneously, namely the source-relay capacity and the overall TDMA channel capacity. In comparing to the point-to-point channel (which uses the same total power), the benefit for utilizing the TDMA relay with BPSK modulation is assessed at as much as 5 dB.

V. CONCLUSION

This paper has developed a practical cooperative coding strategy for the half-duplex relay channel. Achievable rates for all major known relay strategies are quantified, along with the best known upper bound on capacity, for the distance-half geometry. First, the decode-and-forward strategy is observed to dominate the alternative cooperative strategies. Then, it is shown that the rate of the TDMA relay code closely approximates the capacity approaching decode-and-forward code rate. Thus, simple variations on rate-compatible codes can achieve significant cooperative coding gains, with respect to a point-to-point communication channel, without the complexity of the capacity approaching code.

In designing the source and relay channel encoders, the flexibility to address a variety of channel conditions should be emphasized, thereby maximizing their utility. This characteristic would be addressed in a practical system by the use of a granular rate-adaptive source encoder in combination with

[1]See [2] for theoretic results for different relay distances.

jointly optimized rate-compatible code books at the relay. Ideally, the source target rate can be selected specifically for the source-destination and relay-destination channel realizations. To efficiently handle dynamic fading environments such as those found in mobile cellular systems, the relay code books must also address a wide variety of rate requirements.

APPENDIX A
COMPRESS-AND-FORWARD ACHIEVABLE RATE

The compress-and-forward achievable rate is based on source coding results from [5] and [6]. Rate-distortion with side-information at the decoder applies to the relay compression problem, where the destination receives correlated side-information via the direct received source symbol.

In the compress-and-forward strategy, the source message is split into two parts: W_r, representing R_r bits, communicated using the relay, and W_d, representing R_d bits, communicated direct to the destination. The total rate is given by a sum of the component rates $R_{CF} = R_r + R_d$.

In compress-and-forward, the relay does not decode the received symbol, but rather compresses it and sends a quantized version to the destination via a reliable channel (the relay-destination link). In the following achievable rate, the destination is assumed to decode the relay symbol W_r treating the received source symbol W_d as noise. The quantizer rate, R_Q, is then upper bounded by the capacity of the relay-destination link,

$$R_Q \leq (1-\alpha)C\left(\frac{c_3^2\delta E_{MA}}{1+c_2^2(1-\delta)E_{MA}}\right). \tag{19}$$

The distortion corresponding to the relay quantizer rate above is given by the rate-distortion function from source coding theory. Assuming that no additional information is utilized by the decoder when estimating the source symbol, the classical rate-distortion function for a memoryless Gaussian source (which here represents the relay received symbol Y_R) applies [5], where

$$D \geq \frac{c_1^2 E_{BC}+1}{2^{R_Q/\alpha}}, \tag{20}$$

and $D = E(|Y_R - \hat{Y_R}|^2)$.

The received symbols are then processed using MRC, yielding the rate

$$R_r \leq \alpha C\left((c_2^2 + \frac{c_1^2}{1+D})E_{BC}\right). \tag{21}$$

However, a lower quantization distortion level can be achieved on the relay-destination link by noting that the destination BC mode received symbol is correlated with the relay received symbol and can play the role of side-information for the destination decoder. Thus, the receiver side-information, Y_1, in addition to the relay transmitted symbol, X_R, is used to estimate the relay received symbol Y_R. The received symbols Y_1 and $\hat{Y_R}$ are de-correlated prior to estimating the relay symbol W_r, as follows.

The rate-distortion function with jointly Gaussian side-information at the decoder is given by Wyner [6]. Namely, given the relay received symbol

$$Y_R = c_1 X_1 + N_R, \tag{22}$$

the decoder side-information

$$Y_1 = c_2 X_1 + N_1, \tag{23}$$

and "test channel" output

$$X_R = \frac{A-D}{A}(Y_R + N_Q), \tag{24}$$

then

$$D \geq \frac{A}{2^{R_Q/\alpha}}, \tag{25}$$

where $N_Q \sim \mathcal{N}(0, \Sigma_Q)$ is uncorrelated quantization noise with variance $\Sigma_Q = \frac{DA}{A-D}$,

$$A = \Sigma_R - \frac{\Sigma_{1R}^2}{\Sigma_1} \tag{26}$$

is the conditional variance of Y_R given Y_1, and

$$\hat{Y_R} = X_R + \frac{D}{A}\frac{\Sigma_{1R}}{\Sigma_1}Y_1 \tag{27}$$

is the decoder estimate of the relay received symbol, where $\Sigma_R = E(|Y_R|^2)$, $\Sigma_1 = E(|Y_1|^2)$, $\Sigma_{1R} = E(Y_1 Y_R^*)$, and $D = E(|Y_R - \hat{Y_R}|^2)$.

The destination receiver then uses the received symbols Y_1 and $\hat{Y_R}$ to estimate the relay symbol W_r. The MRC rule requires conditional independence of the received symbols given the source symbol and is therefore applied to Y_1 and X_R, yielding the rate

$$R_r' \leq \alpha C\left((c_2^2 + \frac{c_1^2}{1+\Sigma_Q})E_{BC}\right). \tag{28}$$

Finally, the direct symbol W_d is decoded assuming

$$R_d \leq (1-\alpha)C\left(c_2^2(1-\delta)E_{MA}\right), \tag{29}$$

and the compress-and-forward rate

$$R_{CF} = \max\{R_r, R_r'\} + R_d \tag{30}$$

is achievable and is maximized by optimizing the parameters α, β, and δ.

APPENDIX B
LDPC CODE OPTIMIZATION FOR TDMA RELAY

The LDPC parity matrices are optimized using an adaptation of EXIT chart techniques [9]. EXIT charts are an approximation to the Density Evolution (DE) algorithm for analyzing irregular LDPC degree distributions [10]. In EXIT charts, the decoder message densities, which are modeled explicitly in DE, are modeled using a single-parameter family of densities. For this reason, EXIT chart based optimization is more scalable for multi-point communication channels than DE based analysis. A summary of the technique adapted for the TDMA coding framework is provided here.

The difference between a standard EXIT chart optimization and the one used here is (1) the parity matrices developed for the source and relay encoders are constrained to be compatible (i.e. representing the same information bits) and (2) the EXIT chart describing the relay parity matrix is matched to the TDMA channel (mixture of AWGN channels) rather than a point-to-point channel. In this framework, the time- and power-sharing parameters are set to their optimal value as prescribed by the rate maximization for the given modulation alphabet.

For a mixture of AWGN channels, the variable nodes are parameterized by their channel SNR, in addition to the usual variable degree. Thus, let $A(d,s)$ denote the variable-node EXIT function for a degree d variable node with channel SNR s. Further, the degree distribution for variable nodes with channel s is given by $p_v(d,s)$, where $\sum_d p_v(d,s) = 1$. The channel mixture distribution, representing the fraction of code bits with channel s, is written as $p(s)$, where $\sum_s p(s) = 1$. The check-node EXIT functions follow the standard definition, where $B(d)$ denotes the EXIT function for a check node of degree d.

Using the above notation the EXIT chart optimization is performed as follows. As in [8], a base-code parity matrix is constructed using a standard irregular code construction technique. Note that this code will approach the source-relay link capacity. Then, the check-node degree distribution, $p_c(d)$, of the extension-code parity matrix (used by the relay) is obtained by concentrating the average check-node degree near its optimal value as prescribed by the DE algorithm (this can be found online for arbitrary code rates at [11]), while maintaining compatibility with the source parity matrix [8]. This yields the following overall check-node EXIT function: $B_{opt} = \sum_d p_c(d)B(d)$. The extension code parity matrix variable degree distribution is then obtained via least squares curve fitting, as

$$p_{v,opt}(d,s) = \arg\min_{p_v(d,s)} \left\| \sum_{s'} p(s') \sum_{d'} p_v(d',s')A(d',s') - B_{opt}^{-1} \right\|^2, \qquad (31)$$

such that $B_{opt}^{-1} \leq \sum_s p(s) \sum_d p_v(d,s)A(d,s)$.

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