Lecture 5

Laplace transform

Let $s = \sigma + i\omega$ denote a complex number and let $x(t)$ denote a piece-wise continuous function for which $|x(t)| \le M e^{at}$ for some constants M, a.

The Laplace transform is an operator $\mathcal{L}$ that assigns a function $X(s)$ to $x(t)$, where

$$\mathcal{L}[x(t)] \triangleq X(s) \triangleq \int_0^\infty x(t) e^{-st}\, dt,$$

(assume integral includes the point at zero)

If $Re(s) > a$, then

$$\int_0^\infty |x(t) e^{-st}|\, dt = \int_0^\infty |x(t)|\, e^{-\sigma t}\, dt \le \int_0^\infty M e^{-(\sigma - a)t}\, dt < \infty.$$

Hence, $F(s)$ is convergent for any s s.t. $Re(s) > a$.

The Inverse Laplace transform is given by $x(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(s) e^{st}\, ds$ where $c > a$.

Prop. 1: If $x(t)$ has derivative $\dot{x}(t)$, and $\mathcal{L}x$ and $\mathcal{L}\dot{x}$ exist, then: $\mathcal{L}[\dot{x}(t)] = s\mathcal{L}[x(t)] - x(0)$.

pf:
$$\mathcal{L}[\dot{x}(t)] = \int_0^\infty \dot{x}(t) e^{-st}\, dt = \int_0^\infty e^{-st}\, dx(t)$$
$$= e^{-st} x(t)\Big|_0^\infty + s\int_0^\infty x(t) e^{-st}\, dt$$
$$= -x(0) + sX(s).$$

Corollary: $\mathcal{L}[\ddot{x}(t)] = s^2 X(s) - s x(0) - \dot{x}(0)$.

pf: HW

Prop. 2: If $x(t) = e^{-at}$, then $X(s) = \frac{1}{s+a}$.

pf:
$$X(s) = \int_0^\infty e^{-at} e^{-st}\, dt = \int_0^\infty e^{-(s+a)t}\, dt$$
$$= \frac{1}{s+a}, \quad Re(s) > -a.$$

Let $x(t)=0 \ \forall t<0$ and $y(t)=0 \ \forall t<0$.

Thm: If $X(s) = \mathcal{L}[x(t)]$, $Re(s)>\sigma_x$, and $Y(s)=\mathcal{L}[y(t)]$, $Re(s)>\sigma_y$, and $z(t) = x(t) * y(t)$, then $\mathcal{L}[z(t)] = X(s)\,Y(s)$, $Re(s) > \max(\sigma_x,\sigma_y)$.

pf: $z(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$

$$\mathcal{L}[z(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau \; e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} y(t-\tau)\, e^{-st}\, dt\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-s\tau} \int_{-\infty}^{\infty} y(t)\, e^{-st}\, dt\, d\tau$$

$$= Y(s) \cdot \int_{-\infty}^{\infty} x(\tau)\, e^{-s\tau}\, d\tau \qquad Re(s) > \sigma_y$$

$$= X(s)\, Y(s) \qquad Re(s) > \max(\sigma_x, \sigma_y)$$

(left-)

<u>Defn</u>: The right-limit of a function $X(t)$ at t_0 equals α if for any $\varepsilon > 0$ there exists $\delta > 0$ s.t.

$$|f(t) - \alpha| < \varepsilon \quad \text{for } t \in (t_0, t_0+\delta) \qquad (t \in (t_0-\delta, t_0)),$$

and is written $\alpha = \lim_{t\to t_0^+} f(t)$. $\left(\lim_{t\to t_0^-} f(t)\right)$. If no such α exists then the limit of $f(t)$ at t_0 does not exist.

<u>Thm</u>: Initial value thm:

Suppose $\lim_{t\to 0^+} f(t) = \alpha$, and $F(s)$ exists.

Then $\lim_{\sigma\to\infty} \sigma F(\sigma) = \alpha$.

Defn: Limit at infinity
$\lim_{t\to\infty} f(t) = \alpha$ iff $\forall \varepsilon > 0$ $\exists M \in \mathbb{R}$ s.t. $|f(t) - \alpha| < \varepsilon$ $\forall t > M$.

pf: $\int_0^\infty e^{-\sigma t}\,dt = \frac{1}{\sigma}$ $\Rightarrow$

$$|\sigma F(\sigma) - \alpha| = \left|\sigma\int_0^\infty f(t)e^{-\sigma t}dt - \sigma\int_0^\infty \alpha e^{-\sigma t}dt\right|$$

$$\leq \sigma\int_0^\infty |f(t)-\alpha|e^{-\sigma t}dt = \underbrace{\sigma\int_0^y |f(t)-\alpha|e^{-\sigma t}dt}_{(1)} + \underbrace{\sigma\int_y^\infty |f(t)-\alpha|e^{-\sigma t}dt}_{(2)}$$

choose $y > 0$ s.t. $|f(t)-\alpha| < \frac{\varepsilon}{2}$, $t \in (0,y)$ $\Rightarrow$

$$(1) = \sigma\int_0^y |f(t)-\alpha|e^{-\sigma t}dt < \frac{\varepsilon}{2}\sigma\int_0^y e^{-\sigma t}dt = \frac{\varepsilon}{2}\left(1-e^{-\sigma y}\right) < \frac{\varepsilon}{2}, \ \sigma > 0.$$

choose $c \in \mathbb{R}$ s.t. $|F(c)| < \infty$,

$$(2) = \sigma\int_y^\infty |f(t)-\alpha|e^{-\sigma t}dt = \sigma\int_y^\infty e^{-(\sigma-c)t}|f(t)-\alpha|e^{-ct}dt \leq$$

$$\leq \sigma e^{-(\sigma-c)y}\cdot K \quad \left(\text{where } K = \int_y^\infty |f(t)-\alpha|e^{-ct}dt\right)$$

$$\lim_{\sigma\to\infty} K\sigma e^{-(\sigma-c)y} = \lim_{\sigma\to\infty} \frac{K}{ye^{(\sigma-c)y}} = 0 \Rightarrow \exists M \in \mathbb{R} \text{ s.t. } (2) < \frac{\varepsilon}{2} \ \forall \sigma > M$$

$$\Rightarrow |\sigma F(\sigma) - \alpha| < \varepsilon \ \forall \sigma > M \Rightarrow \lim_{\sigma\to\infty} \sigma F(\sigma) = \alpha$$

Defn. The Dirac delta function $\delta(t)$ is a "generalized function" or "distribution" that satisfies the property:

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(\tau - t)\,d\tau = x(t). \qquad \text{"sifting integral"}$$

Properties: (i) $\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$

pf: set $x(t)=1$ in sifting integral

(iii) Let $1(t)$ denote the unit-step function, where: $1(t) = \begin{cases} 0, & t<0 \\ 1/2, & t=0 \\ 1, & t>0 \end{cases}$.

Then $\frac{d}{dt} 1(t) = \delta(t)$, "the generalized derivative of $1(t)$"

pf: Let $x(\tau) = 1(t-\tau)$.

Then $1(t) = x(0) = \int_{-\infty}^{\infty} x(\tau)\,\delta(\tau)\,d\tau = \int_{-\infty}^{t} \delta(\tau)\,d\tau$

$\Rightarrow \frac{d}{dt} 1(t) = \delta(t)$.

[since $\frac{d}{dt} \int_{a(t)}^{b(t)} f(t,\tau)\,d\tau = f(t,a)\cdot\dot{a} - f(t,b)\cdot\dot{b} + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(t,\tau)\,d\tau$

(ii) $\delta(-x) = \delta(x)$

pf: $\int_{-\infty}^{\infty} f(x)\,\delta(-x)\,dx = \int_{-\infty}^{\infty} f(-y)\,\delta(y)\,dy = f(0)$.

(iv) $\delta(x) = 0 \quad \forall x \neq 0$.

pf: assume $x>0$.

$$\delta(x) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x) - u(x)}{\Delta x} = 0.$$

(v) $\mathcal{L}[\delta(t)] = 1$.

pf: $\int_0^{\infty} \delta(t)\,e^{-st}\,dt = 1$. by sifting property

(time-varying)

Defn: An system P is said to be linear if

$$P(c_1 u_1 + c_2 u_2) = c_1 P u_1 + c_2 P u_2 ,$$ where

$P u_i$ is the output corresponding the input u_i, $i=1,2$, and c_1, c_2 are constants.

Defn: The impulse response $h(t,\tau)$ of a linear system P is defined as the output at time t due to a delta-function input at time τ.

Note: δ-function can be thought of as the limit of a sequence of impulsive functions, e.g.

$$P_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{o.w} \end{cases}$$

Then if f is continuous at zero

$$f(0) = \int_{-\infty}^{\infty} \delta(t) f(t)\, dt = \lim_{\Delta \to 0} \int P_\Delta(t) f(t)\, dt$$

pf: HW.

Response of a linear system P to an arbitrary input $u(t)$ [heuristic development]

Let $P\left[\delta(t-\tau)\right] \triangleq h(t,\tau)$.

$$P\left[u(n\Delta)\cdot\Delta\cdot\delta(t-n\Delta)\right] = u(n\Delta)\cdot\Delta\cdot h(t,n\Delta)$$ assume $h(t,n\Delta)$ are bounded

$$P\left[\sum_{n=-\infty}^{\infty} u(n\Delta)\cdot\Delta\,\delta(t-n\Delta)\right] = \sum_{n=-\infty}^{\infty} u(n\Delta)\cdot\Delta\cdot h(t,n\Delta)$$

Letting $\Delta\to 0$: $P[u(t)] =$

$$P\left[\int_{-\infty}^{\infty} u(\tau)\,\delta(t-\tau)\,d\tau\right] = \int_{-\infty}^{\infty} u(\tau)\,h(t,\tau)\,d\tau \triangleq y(t)$$ "super-position integral"

$y(t)$ is the response to input $u(t)$ at time t.

Defn: A linear system is time-invariant if for any α:

$$h(t,\tau) = h(t+\alpha, \tau+\alpha).$$

The impulse response of a time-invariant linear system depends only on the time difference between when the impulse is applied and when the output is measured. For time-invariant system the impulse resp. is written as $h(t) \triangleq h(t,0)$, i.e the response at time t to an impulse at time zero.

Hence, $y(t) = \int_{-\infty}^{\infty} u(\tau)\,h(t-\tau)\,d\tau$, the output is the convolution of the input and the impulse resp.

Defn: A linear system is causal iff. $h(t,\tau)=0$ for any $t<\tau$. I.e. the response to an impulse at time τ is non-zero only for time $t \geq \tau$.

For linear-time-invariant (LTI) systems, causality implies $h(t)=0,\ t<0$.

Hence,
$$y(t) = \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau$$
$$= \int_{-\infty}^{t} u(\tau)\, h(t-\tau)\, d\tau.$$

If $u(t)=0$ for $t<0$, then $y(t) = \int_0^t u(\tau)\, h(t-\tau)\, d\tau$.

Defn The transfer function $H(s)$ of an L.T.I. system is defined as the Laplace transform of the impulse response:

$$H(s) = \int_0^{\infty} h(t)\, e^{-st}\, dt .$$

Thm: If e^{st} is input to a causal LTI system with transfer function $H(s)$, the output is $y(t) = H(s)e^{st}$.

pf:

$$y(t) = \int_{-\infty}^{\infty} e^{s\tau} h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau = e^{st} \int_0^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

$$= e^{st} H(s) .$$

Example: Inhomogeneous linear differential equation of order two, with constant coefficients:

$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 3\dot{u}(t) + 5u(t)$, where $\dot{y}(0) = y(0) = 0$

and $u(t) = \delta(t)$. and $u(0) = 0$.

Soln: $s^2 Y(s) + 3s Y(s) + 2Y(s) = 3s U(s) + 5U(s)$

$$Y(s) = \frac{3s+5}{s^2+3s+2} U(s) \qquad \left[\text{i.e. } H(s) = \frac{3s+5}{s^2+3s+2} \right]$$

$$\frac{3s+5}{s^2+3s+2} = \frac{1}{(s+2)} + \frac{2}{(s+1)}$$

$$\Rightarrow Y(s) = \left[\frac{1}{s+2} + \frac{2}{s+1} \right] \qquad [\, U(s) = 1 \,]$$

$$\Rightarrow y(t) = e^{-2t} + 2e^{-t}, \quad t \geq 0.$$