<u>Discussion</u> We defined the determinant of A as

$$\det(A) = \sum_{k=1}^{n} [A]_{1k} C_{1k}$$ "cofactor expansion across the first row"

and derived several properties. It follows from the properties that the determinant can be computed by evaluating the cofactor expansion across any row or column, ie.

$$\det(A) = \sum_{k=1}^{n} [A]_{ik} C_{ik}$$ "cofactor expansion across ith row"

$$= \sum_{k=1}^{n} [A]_{kj} C_{kj}$$ "cofactor expansion across jth column"

pf exercise (hint: by transpose property it follows that if any two rows of A are interchanged then the determinant changes sign. And from row interchange property it follows that a cofactor expansion accross ith row equals the cofactor expansion across the 1^{st} row.

Defn: A vector space V is a set of elements $\{x, y, \dots\}$ with addition and multiplication by a scalar in $\mathbb{F}$ where:

1. $(x+y) \in V$, $x, y \in V$
2. $\alpha x \in V$, $x \in V$, $\alpha \in \mathbb{F}$
3. $\exists\, \underline{0} \in V$ s.t. $0x = \underline{0}$ $\forall x \in V$
4. $1x = x$, $\forall x \in V$
5. $x+y = y+x$
6. $(x+y)+z = x+(y+z)$
7. $\alpha(\beta x) = (\alpha\beta)x$, $\alpha, \beta \in \mathbb{F}$
8. $(\alpha+\beta)x = \alpha x + \beta x$
9. $\alpha(x+y) = \alpha x + \alpha y$

Defn: A set of vectors $\{\underline{x}_j\}$, $j=1,\dots,p$ are linearly dependent iff $\sum_{j=1}^{p} c_j \underline{x}_j = 0 \Rightarrow c_j = 0, j=1,\dots,p$. Otherwise $\{x_j\}$ are said to be linearly independent.

Defn A set of vectors $\{x_1, \dots, x_n\}$, $\underline{x}_j \in V$, $j=1,\dots,n$, span the vector space V iff $\left\{ \sum_{j=1}^{n} c_j \underline{x}_j : c_j \in \mathbb{F}, j=1,\dots,n \right\} = V$.

Defn A basis for the vector space V is any set of linearly independent vectors $\{\underline{x}_j\}$, $j=1,\dots,n$ s.t. the span of $\{\underline{x}_j\}$ equals V.

Defn: The dimension of a vector space V, denoted $\dim(V)$, is an integer n for which there exists n-linearly independent vectors and any set of $(n+1)$ vectors is linearly dependent.

E.g. The $n \times 1$ column "n-tuples" matrices with elements from $\mathbb{F}$ form an n-dim. vector space, spanned by $\{e_1, \dots, e_n\}$, where $e_i = [0 \cdots \overset{i^{th}}{1} \cdots 0]^T$.

In this course we always assume this vector space.

The inner product on the n-tuple vector space with $x = [x_1, \dots, x_n]$ and $y = [y_1, \dots, y_n]$ is defined as

$$(x, y) = y^* x = \sum_{k=1}^{n} x_k \overline{y_k} .$$

If x and y are real, then $(x,y) = y^T x$.

Defn: The magnitude or norm of a vector $x = [x_1, \dots, x_n]$ is

$$\|x\| = \sqrt{(x,x)} = \sqrt{\sum_{k=1}^{n} x_k \overline{x_k}} .$$

Defn: The vectors x and y are orthogonal iff $(x, y) = 0$.

Defn: An orthonormal set of vectors $\{x_i\}$, $i = 1, \dots, p$, has the property $(x_i, x_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & \text{o.w.} \end{cases}$

Defn: An $n \times n$ orthogonal matrix Q is defined as a matrix for which $Q^T Q = I$. I.e. $Q^{-1} = Q^T$.

Defn: An $n \times n$ unitary matrix U is such that $U^* U = I$. I.e. $U^{-1} = U^*$.

Defn: Inner product (x, y) on a vector space V

(a). $(y, x) = \overline{(x, y)}$, $x, y \in V$

(b). $(x+y, z) = (x, z) + (y, z)$, $x, y, z \in V$

(c) $(\alpha x, y) = \alpha (x, y)$, $x, y \in V$, $\alpha \in \mathbb{F}$

(d) $(x, x) \geq 0$, $x \in V$

(e) $(x, x) = 0$ iff $x = 0$

Defn The classical adjoint or adjugate $\mathrm{Adj}(A)$ of an $n\times n$ matrix A is a matrix comprised of the cofactors of A, where $[\mathrm{Adj}(A)]_{ij} = C_{ji}(A)$, $1\le i\le n$, $1\le j\le n$.

Thm: If $\det(A)\neq 0$, then:

$$A^{-1} = \frac{\mathrm{Adj}(A)}{\det(A)}.$$

pf: Consider the product of $\mathrm{Adj}(A)$ and A:

$$[\mathrm{Adj}(A)\cdot A]_{ij} = \sum_{k=1}^{n} [\mathrm{Adj}(A)]_{ik}\,[A]_{kj}$$

$$= \sum_{k=1}^{n} [A]_{kj}\, C_{ki} = (*)$$

if $i=j$, then $(*) = \det(A)$ (cofactor expansion across ith column)

if $i\neq j$, then $(*) = \det\left(\left[\underline{a}_1,\dots,\underline{a}_{i-1},\underline{a}_j,\underline{a}_{i+1}\dots\underline{a}_n\right]\right) = 0$,

(repeated column)

where $\underline{a}_k$ denotes the kth column of A, $k=1,\dots,n$.

Hence, $\mathrm{Adj}(A)\cdot A = \det(A)\cdot I$, and $\det(A)\neq 0 \Rightarrow$

$$\frac{\mathrm{Adj}(A)}{\det(A)}\cdot A = I, \quad \text{i.e. } A^{-1} = \frac{\mathrm{Adj}(A)}{\det(A)}.$$ (since A^{-1} is unique)

Thm: A^{-1} exists iff $\det(A)\neq 0$.

pf: $\Rightarrow$ Assume A^{-1} exists, but $\det(A)=0$.

Then $\det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1}) = 0$.

Contradicts $\det(I)=1 \Rightarrow \det(A)\neq 0$.

$\Leftarrow$ Assume $\det(A)\neq 0$. Then:

$$A^{-1} = \frac{\mathrm{Adj}(A)}{\det(A)}.$$

$\det(A)=0$: "A is singular"
$\det(A)\neq 0$: "A is nonsingular"

Thm: Cramer's rule.

Suppose $Ax = b$, where A is a nonsingular ($\det A \neq 0$) $n \times n$ matrix with known elements, b is an $n \times 1$ vector with known elements, and x is an $n \times 1$ vector with unknown elements. Then,

$$[x]_{j1} = \frac{\det([\underline{a}_1, \ldots, \underline{a}_{j-1}, \underline{b}, \underline{a}_{j+1}, \ldots, \underline{a}_n])}{\det(A)}$$

pf: $$x = A^{-1} b = \frac{\mathrm{Adj}(A)}{\det(A)} b$$

where $$[\mathrm{Adj}(A)\, b]_{j1} = \sum_{k=1}^{n} [\mathrm{Adj}(A)]_{jk} [b]_{k1}$$

$$= \sum_{k=1}^{n} [b]_{k1} C_{kj} = \det([\underline{a}_1, \ldots, \underline{b}, \ldots, \underline{a}_n]).$$

The claim follows upon dividing by $\det(A)$.

<u>Defn</u>: Let A denote an $m\times n$ matrix. The <u>column space</u> or range of A is defined as

$$R(A) = \{ Ax : x \in \mathbb{F}^n \}$$

$R(A)$ is a vector space, or a "subspace" of $\mathbb{F}^m$.

<u>Defn</u>: The <u>null space</u> of A $m\times n$ is defined as

$$\mathcal{N}(A) = \{ x \in \mathbb{F}^n : Ax = 0 \}$$

$\mathcal{N}(A)$ is a subspace of $\mathbb{F}^n$

<u>Defn</u>: The rank of A, denoted $\operatorname{rank}(A)$, is defined as the number of linearly independent columns of A.

<u>Thm</u>: $\operatorname{rank}(A) = \dim(R(A))$. <u>pf</u>: exercise

Let A denote an $n \times n$ matrix.

Thm: $\det(A) = 0$ iff cols. of A are lin. dep.

$\Rightarrow$] Assume $\det(A) = 0$

1] If $[A]_{k1} = 0$, $k = 1, \dots, n$, then $\det(A) = 0$ and cols lin. dep. [end alg.]

Else: let $A_1 = E_1 A U_1$

where $A_1 = \begin{bmatrix} x & 0 & \cdots & 0 \\ x & x & \cdots & x \\ & & & \\ x & x & \cdots & x \end{bmatrix}$ $E_1 = \begin{array}{c|c} & \begin{array}{cc} j & i \end{array} \\ \begin{array}{c} j \\ i \end{array} & \begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ 1 & & 1 & \\ & & & 1 \end{bmatrix} \end{array}$ $\Rightarrow \det(E_1) = -1$ $U_1 = \begin{bmatrix} 1 & x & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & & & \\ 0 & \cdots & & & 1 \end{bmatrix} \Rightarrow \det(U_1) = 1$

or $E_1 = I \Rightarrow \det(E_1) = 1$

$$\det(A_1) = \det(E_1)\det(A)\det(U_1) = (-1)^{k_1} \cdot \det(A), \quad k_1 \in \{0,1\}$$

$\vdots$

j] If $[A_{j-1}]_{k,j} = 0$, $k = 1, \dots, n$, then $\det(A_{j-1}) = 0 \Rightarrow \begin{cases} \det(A) = 0 \text{ and} \\ \text{cols. } A \text{ lin. dep} \end{cases}$ [end alg.]

$(\exists [c_k]: \underline{a}_j = \sum_{k=1}^{j-1} c_k \underline{a}_k)$

Else: Let $A_j = E_j A_{j-1} U_j$

where $A_j = \begin{bmatrix} x & 0 & & 0 & 0 & \cdots & 0 \\ x & x & & & & & \vdots \\ & & x & 0 & 0 & & 0 \\ & & x & x & x & & \vdots \\ x & \cdots & & & & & x \end{bmatrix}$ (jth), E_j = perm. matrix or $E_j = I$, $U_j = \begin{bmatrix} 1 & 0 & & x & 0 \\ & & & \vdots & \\ 0 & & 1 & x & \\ & & & 1 & 0 \ 0 \\ 0 & & & & 1 \end{bmatrix}$,

and $\det(A_j) = \det(E_j)\det(A_{j-1})\det(U_j) = (-1)^{k_j} \det(A)$

$\vdots$

n] If $[A_{n-1}]_{kn} = 0$, $k = 1, \dots, n$, then $\det(A_{n-1}) = 0 \Rightarrow \det(A) = 0$ and cols. A are lin. dep. [end alg.]

Else, $\det(A_{n-1}) = \prod_{j=1}^{n} [A_{n-1}]_{jj} \neq 0 \Rightarrow \begin{cases} 1. \det(A) \neq 0. \\ 2. \text{ cols. of } A \text{ are lin. ind.} \end{cases}$

(pf: [exercise] show $A_{n-1} x = 0 \Rightarrow x = 0$.)

Hence, cols. of A are lin. dep. since could not have arrived at last step

$\Leftarrow$] Assume cols. of A are lin. dep.

$$\Rightarrow \exists k \in [1, \dots, n] \text{ and } \{c_j\}, j \neq k \text{ s.t. } \sum_{j \neq k} c_j \underline{a}_j = \underline{a}_k$$

$$\Rightarrow \det(A) = \det\left(\left[a_1 \cdots \sum_{j \neq k} c_j a_j \cdots a_n\right]\right) = \sum_{j \neq k} c_j \det([a_1 \cdots \overset{k\text{th}}{a_j} \cdots a_n]) = 0.$$

corollary] $\det(A) \neq 0$ iff $\operatorname{rank}(A) = n$.

Thm: If A is $n\times n$, with $\det(A)\neq 0$, and B is $m\times n$, then $\mathrm{rank}(BA)=\mathrm{rank}(B)$.

pf: Let $e_1,\dots,e_n$ denote a basis for $\mathbb{F}^n$. Then $Ae_1,\dots,Ae_n$ are lin. ind., for suppose not, then: $\exists\ \{c_j\},\ j=1,\dots,n$ s.t.

$$\sum_{j=1}^n c_j A e_j = 0 \Rightarrow A\sum_{j=1}^n c_j e_j = 0 \Rightarrow \text{cols. } A \text{ lin. dep. which contradicts } \det(A)\neq 0.$$

So $\{Ae_j\}$ is basis for $\mathbb{F}^n$ and: $\{Ax : x\in\mathbb{F}^n\} = \{A\sum_{j=1}^n \alpha_j e_j : \alpha_j\in\mathbb{F},\ j=1,\dots,n\}$

$$= \{\sum_{j=1}^n \alpha_j A e_j : \alpha_j\in\mathbb{F},\ j=1,\dots,n\} = \mathbb{F}^n$$

Hence, $R(BA) = \{BAx : x\in\mathbb{F}^n\} = \{By : y\in\mathbb{F}^n\} = R(B)$

$\Rightarrow \mathrm{rank}(BA) = \mathrm{rank}(B)$.

Thm: Let A denote an $m\times n$ matrix. Then $\mathrm{rank}(A) = \mathrm{rank}(A^T)$.

pf: HW.

Thm: The fundamental theorem of algebra.

Let $p(s)$ denote an nth order polynomial with complex coefficients defined on the complex plane, with $p(s) = s^n + p_1 s^{n-1} + \dots + p_n$, $s\in\mathbb{C}$.

Then $\exists\ \lambda\in\mathbb{C}$ s.t. $p(\lambda)=0$.

Corollary 1. $\exists\ \lambda_j\in\mathbb{C}$, $j=1,\dots,n$ s.t. $p(s) = \prod_{j=1}^n (s-\lambda_j)$.

Corollary 2. If $p_j\in\mathbb{R}$, $j=1,\dots,n$, then: $p(\lambda)=0 \Leftrightarrow p(\bar\lambda)=0$.

pf: $\overline{p(s)} = p(\bar s) = \prod_{j=1}^n (\bar s-\lambda_j) = \overline{\prod_{j=1}^n (s-\bar\lambda_j)}$

$$\Rightarrow p(s) = \prod_{j=1}^n (s-\lambda_j) = \prod_{j=1}^n (s-\bar\lambda_j) \Rightarrow p(\lambda_j)=0 \Leftrightarrow p(\bar\lambda_j)=0.$$

<u>Defn</u>: The <u>characteristic</u> <u>polynomial</u> of an $n \times n$ matrix A, denoted by $a(s)$, is defined as $a(s) = \det([sI - A])$.

Note: $a(s)$ is nth order poly. (pf: HW) Let $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n \overset{?}{=} \prod_{j=1}^{n} (s - \lambda_j)$ — fund. thm alg.

<u>Defn</u>: The <u>characteristic</u> <u>values</u> of A are defined as

$$\{\lambda \in \mathbb{C} \text{ s.t. } a(\lambda) = 0\}.$$

Let λ denote a ch. value of A,

$\Rightarrow \det(\lambda I - A) = 0$

$\Rightarrow \exists\, \underline{v} \in \mathbb{F}^n$ s.t. $(\lambda I - A) v = 0$ "v is in the null space of $\lambda I - A$" $\Rightarrow Av = \lambda v$

<u>Defn</u>: A vector $\underline{v}$ in the null space of $\lambda I - A$, $\lambda \in \mathbb{C}$, is defined as a <u>ch.</u> <u>vector</u> of A corresponding to the ch. value λ.

A ch. vector defines a one-dimensional invariant subspace of A, i.e. $\underline{x} \in R(\underline{v}) \Rightarrow \underline{x} = c\underline{v} \Rightarrow Ax = cAv = c\lambda v = \lambda x \in R(\underline{v})$

<u>Defn</u>: Let A and B denote $n \times n$ matrices. The matrix A is <u>similar</u> to B if there exists an invertible $n \times n$ matrix T such that $B = T^{-1} A T$.

<u>Thm</u>: If A has n linearly independent ch. vectors, then A is similar to a diagonal matrix.

pf: Let $\{\lambda_j\}$ denote the ch. values corresp. to the ch. vectors $\{\underline{v}_j\}$, $j=1,\dots,n$.

Then: $A\underline{v}_j = \lambda_j \underline{v}_j \Rightarrow AT = T\Lambda$, where $T = [\underline{v}_1, \dots, \underline{v}_n]$ and

$\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$. $Det(T) \neq 0 \Rightarrow \Lambda = T^{-1}AT$.

<u>Note</u>: If $\Lambda = T^{-1}AT = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$, then

$$A = T\Lambda T^{-1} = \sum_{j=1}^{n} \lambda_j \underline{v}_j \underline{w}_j^T, \quad \text{where}$$

$T = [\underline{v}_1, \dots, \underline{v}_n]$ and $T^{-1} = [\underline{w}_1, \dots, \underline{w}_n]^T$