Lecture 1

<u>Defn</u>: An $m \times n$ <u>matrix</u> over the field $\mathbb{F}$ (e.g. $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=GF(q)$) is a rectangular array of numbers from $\mathbb{F}$.

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{m1} & & a_{mn} \end{bmatrix}$ denote an $m \times n$ matrix.

The numbers that constitue A are called its elements and the jth element of the ith row can be written as $a_{ij} = [A]_{ij}$. The <u>dimension</u> of A is $m \times n$. If A is an $n \times n$ matrix, it is said to have order n.

Let A and B denote $m \times n$ matrices and let C denote an $n \times p$ matrix. The following algebreic operations are defined:

1. Multiplication by a scalar

$$[\alpha A]_{ij} = \alpha [A]_{ij}, \quad 1 \le i \le m, \ 1 \le j \le n$$

2. Matrix addition

$$[A+B]_{ij} = [A]_{ij} + [B]_{ij}, \quad 1 \le i \le m, \ 1 \le j \le n$$

3. Matrix multiplication

$$[AB]_{ij} = \sum_{k=1}^{n} [A]_{ik}[B]_{kj}, \quad 1 \le i \le m, \ 1 \le j \le n$$

Note that the number of columns of A must equal the number of rows of B. In general, $AB \ne BA$, However, if $AB = BA$, A and B are said to commute.

4. The transpose of a matrix is denoted A^T, where

$$[A^T]_{ij} = [A]_{ji}$$

5. Matrix inverse. Let A denote an $n \times n$ matrix, if $\exists$ an $n \times n$ matrix B s.t. $AB = I$, then B is said to be the inverse of A and is denoted A^{-1}.

6. Complex conjugate of a matrix A is denoted $\bar{A}$, where

$$[\bar{A}]_{ij} = \overline{[A]_{ij}} = Re([A]_{ij}) - i\, Im([A_{ij}])$$

7. Conjugate transpose (Hermitian transpose) of A is denoted A^*

where $[A^*]_{ij} = [\bar{A}]_{ji}$.

Properties (1)–(18) follow from the above definitions. (pf: exercise) Let A, B, and C denote matrices of compatible dimensions, and $\alpha, \beta \in \mathbb{F}$.

(1) $A+B = B+A$

(2) $(A+B)+C = A+(B+C)$

(3) $\alpha(A+B) = \alpha A + \beta B$

(4) $(\alpha\beta)A = \alpha(\beta A)$

(5) $A-B = A+(-1)B$

(6) $(AB)C = A(BC)$ associativity of matrix mult.

(7) $(A+B)C = AC+BC$ right distr.

(8) $A(B+C) = AB+AC$ left distr.

(9) $(A+B)^T = A^T + B^T$

(10) $(AB) = B^T A^T$

(11) $(A^*)^* = A$

(12) $(A+B)^* = A^* + B^*$

(13) $(\alpha A)^* = \bar{\alpha} A^*$

(14) $(AB)^* = B^* A^*$

(15) $AA^{-1} = A^{-1}A$

(16) $(AB)^{-1} = B^{-1}A^{-1}$

(17) $(A^T)^{-1} = (A^{-1})^T$

(18) $(A^*)^{-1} = (A^{-1})^*$

<u>Defn</u>

The <u>determinant</u> of a square matrix A is a number in the field $\mathbb{F}$, denoted $\det(A)$ or $|A|$.

The determinant of a 1×1 matrix $A = a$ is a.

The determinant of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is defined as: $\det(A) = ad - bc$.

And, the determinant of an $n \times n$ matrix A, with elements $[A]_{jk} = a_{jk}$, is defined inductively in terms of the determinant of an $(n-1)\times(n-1)$ matrix, as:

$$\det(A) = \sum_{k=1}^{n} [A]_{1k} \, C_{1k},$$

where C_{jk} denotes the (j,k)th <u>cofactor</u> or algebraic complement of the element $[A]_{jk}$, where: $C_{jk} = (-1)^{j+k} M_{jk}$,

and:

$$M_{jk} = \det\left(\begin{bmatrix} a_{11} & \cdots & a_{1(k-1)} & a_{1(k+1)} & \cdots & a_{1n} \\ \vdots & & & & & \\ a_{(j-1)1} & \cdots & a_{(j-1)(k-1)} & a_{(j-1)(k+1)} & & a_{(j-1)n} \\ a_{(j+1)1} & & a_{(j+1)(k-1)} & a_{(j+1)(k+1)} & & a_{(j+1)n} \\ a_{n1} & \cdots & a_{n(k-1)} & a_{n(k+1)} & \cdots & a_{nn} \end{bmatrix} \right), \quad 1 \le j \le n, \quad 1 \le k \le n.$$

M_{jk} is the determinant of the matrix formed from A by deleting the jth row and kth column. M_{jk} is called the minor of the element $[A]_{jk}$

<u>Properties of determinants</u> (0.) $\det(I) = 1$.

(1.) If any two columns of A are interchanged, then the determinant of the resulting matrix is equal to $(-1)|A|$.

pf: (by induction) In the 2x2 case, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then,

$$\det\left(\begin{bmatrix} b & a \\ d & c \end{bmatrix}\right) = bc - ad = -\det(A).$$

Now assume that the property is true for any $(n-1) \times (n-1)$ matrix.

Let A be $n \times n$, with $A = [\underline{a}_1 \cdots \underline{a}_n]$, where $\underline{a}_j$ denotes the jth column of A. ($\underline{a}_j$ is an $n \times 1$ "column" matrix.)

First assume column k is interchanged with column $k+1$. Then,

$$\det\left(\left[\underline{a}_1, \cdots \underline{a}_{k-1}, \underline{a}_{k+1}, \underline{a}_k, \underline{a}_{k+2}, \cdots, \underline{a}_n\right]\right) = \sum_{j=1}^{n} (-1) [A]_{1j} C_{1j}$$

$= -\det(A)$. The general case follows by repeated interchanges of adjacent columns.

(2.) If any column of A is multiplied by a scalar α, the determinant of the resulting matrix is $\alpha|A|$.

pf: (by induction) In the 2x2 case, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$\det\left(\begin{bmatrix} \alpha a & b \\ \alpha c & d \end{bmatrix}\right) = \det\left(\begin{bmatrix} a & \alpha b \\ c & \alpha d \end{bmatrix}\right) = \alpha(ad - bc) = \alpha|A|.$$

Now assume that the property is true for any $(n-1) \times (n-1)$ matrix. And let A be $n \times n$, with $A = [\underline{a}_1, \cdots, \underline{a}_n]$. Then,

$$\det\left(\left[\underline{a}_1, \cdots, \underline{a}_{k-1}, \alpha \underline{a}_k, \underline{a}_{k+1}, \cdots, \underline{a}_n\right]\right) = \sum_{j=1}^{n} \alpha [A]_{1j} C_{1j}$$

$= \alpha \det(A)$.

Property (3.) If any column of A is proportional to another column of A, then $|A| = 0$.

pf: (by induction) In the 2x2 case:

$$\det\left(\begin{bmatrix} \alpha b & b \\ \alpha d & d \end{bmatrix}\right) \underset{\text{by prop.(2)}}{=} \alpha \det\left(\begin{bmatrix} b & b \\ d & d \end{bmatrix}\right) = \alpha(bd - bd) = 0.$$

Now assume that the property is true any $(n-1) \times (n-1)$ matrix.

And let A be $n \times n$, with $A = [\underline{a}_1, \dots, \underline{a}_n]$. Then,

$$\det\left(\left[\underline{a}_1 \cdots \underline{a}_{j-1}\ \alpha \underline{a}_k\ \underline{a}_{j+1}, \cdots \underline{a}_n\right]\right) =$$

$$= \alpha [A]_{1k} (-1)^{1+j} M_{1j} + [A]_{1k} (-1)^{1+k} M_{1k}$$

$$= \alpha [A]_{1k} (-1)^{1+j} M_{1j} + \alpha [A]_{1k} (-1)^{1+k} \tilde{M}_{1k}$$ ← j^{th} column is divided by α

$$= \alpha [A]_{1k} (-1)^{1+j} M_{1j} + \alpha [A_{1k}] \underbrace{(-1)^{1+k} (-1)^{k-j-1}}_{=(-1)^j} M_{1j}$$ ← by prop. 1

$$= 0.$$

<u>property (4.)</u> The determinant is a linear function of a given column when all other columns are fixed.

Let $A = [\underline{a}_1 \cdots \underline{a}_n]$.

Then $\det([\underline{a}_1, \ldots, \underline{a}_{k-1}, \underline{b} + \underline{a}_k, \underline{a}_{k+1}, \ldots \underline{a}_n])$

$= \det(A) + \det([\underline{a}_1, \ldots, \underline{a}_{k-1}, \underline{b}, \underline{a}_{k+1}, \ldots, \underline{a}_n])$

<u>pf:</u> (by induction) In the 2x2 case, with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$\det\left(\begin{bmatrix} a+b_{11} & b \\ c+b_{21} & d \end{bmatrix}\right) = (a+b_{11})d - (c+b_{21})b = (ad-bc) + (b_{11}d - b_{21}b)$

$= \det(A) + \det\left(\begin{bmatrix} b_{11} & b \\ b_{21} & d \end{bmatrix}\right)$

Now assume that the property is true for any $(n-1)\times(n-1)$ matrix.
And let A denote an $n\times n$ matrix with $A = [\underline{a}_1, \ldots, \underline{a}_n]$. Then:

$\det([\underline{a}_1, \ldots, \underline{b} + \underline{a}_k, \ldots, \underline{a}_n]) = \sum_{\substack{j=1 \\ j\neq k}}^{n} [A]_{1j}\{C_{1j}(A) + C_{1j}([\underline{a}_1, \ldots, \underline{b}, \ldots \underline{a}_n])\} + [\underline{b} + \underline{a}_k]_{11} C_{1k}(A)$

where $C_{ij}(A)$ denotes the cofactor of the (i,j)th element of A.

It follows that $\det([\underline{a}_1, \ldots, \underline{b} + \underline{a}_k, \ldots, \underline{a}_n]) = \det(A) + \det([\underline{a}_1, \ldots, \underline{b}, \ldots, \underline{a}_n])$.

<u>Corollary</u>: If a multiple of any column of A is added to another column of A then the determinant of the resulting matrix is unchanged, ie. if $A = [\underline{a}_1, \ldots, \underline{a}_n]$, then

$\det([\underline{a}_1, \ldots, \alpha \underline{a}_j + \underline{a}_k, \ldots, \underline{a}_n]) = \det(A)$, where $1 \le k \le n$, $1 \le j \le n$, $j \neq k$, $\alpha \in \mathbb{F}$.

pf: By column linearity (prop. (4.))

$$\det([\underline{a}_1, \ldots, \alpha \underline{a}_j + \underline{a}_k, \ldots, \underline{a}_n]) = \underbrace{\det([\underline{a}_1, \ldots, \alpha \underline{a}_j, \ldots, \underline{a}_n])}_{=0 \text{ by prop. (3.)}} + \det(A)$$

Hence, $\det([\underline{a}_1, \ldots, \alpha \underline{a}_j + \underline{a}_k, \ldots \underline{a}_n]) = \det(A)$.

Property (5.) Let $\Pi(1,\dots,n)$ denote the set of $n!$ permutations of the integers $(1,\dots,n)$ and let $\mu(p)$ denote the number of interchanges of elements of $p \in \Pi$ that are needed to produce the usual order $(1,\dots,n)$.

Then, $\det(A) = \sum_{p\in\Pi(1,\dots,n)} (-1)^{\mu(p)} \prod_{k=1}^{n} [A]_{kp(k)}$, where $p(k)$ is the kth element of permutation p.

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pf: (by induction) In the 2x2 case, with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

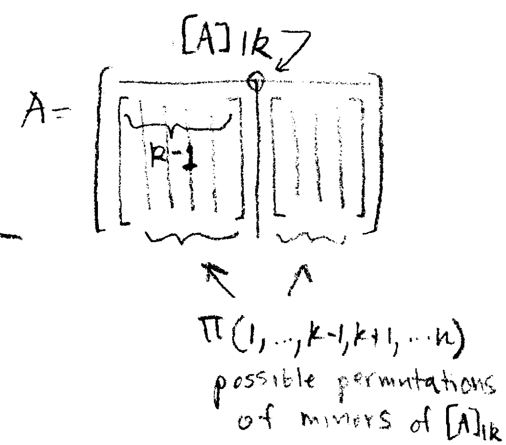
$$\sum_{p\in\{[1\,2],[2\,1]\}} (-1)^{\mu(p)} \prod_{k=1}^{2} [A]_{kp(k)} = (-1)^0 ad + (-1)^1 cd = \det(A).$$

Now assume that the property is true for any $(n-1)\times(n-1)$ matrix. And let A denote an $n\times n$ matrix. Then:

$$\det(A) = \sum_{k=1}^{n} [A]_{1k} C_{1k} = \sum_{k=1}^{n} [A]_{1k} (-1)^{k+1} M_{1k}$$

$$= \sum_{k=1}^{n} [A]_{1k} (-1)^{k+1} \sum_{p\in\Pi(1,\dots,k-1,k+1,\dots,n)} (-1)^{\mu(p)} \prod_{j=1}^{n-1} [A]_{(j+1)p(j)}$$

$$= \sum_{p\in\Pi(1,\dots,n)} (-1)^{\mu(p)} \prod_{j=1}^{n} [A]_{jp(j)} \quad \blacksquare$$



Corollary: [HW]

$$\det(A) = \sum_{p\in\Pi(1,\dots,n)} (-1)^{\mu(p)} \prod_{k=1}^{n} [A]_{p(k)k}.$$

<u>Property (6.)</u> The Determinant of the product of two matrices is the products of the determinants

Let A denote an $n \times n$ matrix and B denote an $n \times n$ matrix.
And let $C = AB$. Then $|C| = |A||B|$.

pf: Denote the elements of B as $[B]_{ij} = b_{ij}$, $1 \le i \le n$, $1 \le j \le n$, and denote the columns of A as $\underline{a}_k$, $1 \le k \le n$.

Then
$$C = \left[\sum_{k_1=1}^{n} b_{k_1 1} \underline{a}_{k_1}, \ldots, \sum_{k_n=1}^{n} b_{k_n n} \underline{a}_{k_n} \right].$$

It follows from the column linearity property (Prop. 4) that:

$$\det(C) = \sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} \det\left(\left[b_{k_1 1} \underline{a}_{k_1}, \ldots, b_{k_n n} \underline{a}_{k_n} \right] \right),$$

and from property (3.) it follows that:

$$\det(C) = \sum_{g \in \Pi(1,\ldots,n)} \det\left(\left[b_{g(1)1} \underline{a}_{g(1)}, \ldots, b_{g(n)n} \underline{a}_{g(n)} \right] \right)$$

and from property (2.) it follows that:

$$\det(C) = \sum_{g \in \Pi(1,\ldots,n)} \left\{ \prod_{j=1}^{n} b_{g(j)j} \right\} \det\left(\left[\underline{a}_{g(1)}, \ldots, \underline{a}_{g(n)} \right] \right)$$

$$= \sum_{g \in \Pi(1,\ldots,n)} \left\{ \prod_{j=1}^{n} b_{g(j)j} \right\} (-1)^{\mu(g)} \det(A)$$

$$= \det(A) \cdot \underbrace{\sum_{g \in \Pi(1,\ldots,n)} (-1)^{\mu(g)} \prod_{j=1}^{n} b_{g(j)j}}_{= \det(B) \text{ by corollary to prop (5.)}}$$

Hence, $\det(C) = \det(A) \cdot \det(B)$.

<u>corollary</u> $\det(A^T) = \det(A)$ pf: HW
and $\det(A^{-1}) = \det(A)^{-1}$.