E6201 Linear Systems - HW11 solution

1] $(sI - A_c)x = b_c$ Cramer's rule:

$$[x]_{j1} = \frac{\begin{vmatrix} s+a_1 & a_2 & \cdots & a_{j-1} & 1 & a_{j+1} & \cdots & a_{n-1} & a_n \\ -1 & s & & & 0 & & & & \\ 0 & -1 & & \ddots & \vdots & & & & \\ & & & & & & & & \\ 0 & 0 & \cdots & & 0 & & & -1 & s \end{vmatrix}}{|sI - A_c|}$$

$$= \frac{1}{a(s)} (-1)^{j-1} \begin{vmatrix} -1 & s & 0 & \cdots & 0 & \overset{j^{th}}{0} & & 0 & 0 \\ 0 & -1 & s & \cdots & s & 0 & & \vdots & \vdots \\ 0 & 0 & -1 & & -1 & 0 & & & \\ & \vdots & & & & s & \cdots & s & 0 \\ & & & & & -1 & \cdots & -1 & s \end{vmatrix}$$

$$= \frac{1}{a(s)} (-1)^{j-1} \cdot (-1)^{j-1} \cdot \begin{vmatrix} s & & \\ -1 & \ddots & \\ & -1 & s \end{vmatrix} = \frac{s^{n-j}}{a(s)}$$

2] $Abc^TAb = c^TAb \cdot Ab \Rightarrow c^TAb$ is ch. value of Abc^T

$c^Tw = 0 \Rightarrow Abc^Tw = 0 \cdot w \Rightarrow 0$ is ch. value of Abc^T w/ multiplicity $n-1$

$\Rightarrow$ ch. values of $I + Abc^T$ are 1 ($n-1$ times) and $1 + c^TAb$

$\Rightarrow \det(I + Abc^T) = 1 + c^TAb$

4] (3.3-1) $\underline{x} \triangleq \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} \Rightarrow \dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} u \triangleq Ax + bu$

$\alpha(s) = (s+1)^2(s+1-j)(s+1+j) = s^4 + 4s^3 + 7s^2 + 6s + 2$

$a(s) = \det(sI - A) = s^2(s^2 - 1)$

$\mathcal{C}(A,b) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -\beta \\ -1 & 0 & -\beta & 0 \end{bmatrix}$ $\quad a_- = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$ $\quad \alpha - \underline{a} = 2 \cdot \begin{bmatrix} 2 \\ 4 \\ 3 \\ 1 \end{bmatrix}$

$(a_- \mathcal{C}^T)^{-1} = \begin{bmatrix} 0 & 1 & 0 & (1-\beta)^{-1} \\ 1 & 0 & (1-\beta)^{-1} & 0 \\ 0 & 0 & 0 & (1-\beta)^{-1} \\ 0 & 0 & (1-\beta)^{-1} & 0 \end{bmatrix}$ $\quad k = (a_- \mathcal{C}^T)^{-1}(\alpha - a)$

$= 2\left[4 + (1-\beta)^{-1} \quad 2 + 3(1-\beta)^{-1} \quad (1-\beta)^{-1} \quad 3(1-\beta)^{-1}\right]^T,$

$\beta \neq 1.$

Kailath, Example 3.3-6

#1. Let $A_c = T^{-1}AT$ $\quad b_c = T^{-1}b$ $\quad c_c^T = c^T T$

$$sI - A + bk^T = sI - TA_cT^{-1} + bk^T$$

$$= T(sI - A_c + T^{-1}bk^TT)T^{-1}$$

$$= T(sI - A_c + b_ck_c^T)T^{-1} \qquad k_c^T = k^TT$$

$\{A, b\}$ controllable $\Rightarrow$ can transform to $\{A_c, b_c, c_c\}$ controller form

$\Rightarrow$ $\{A_c - b_ck_c^T, b_c\}$ is controllable

$\Rightarrow$ $\{A - bk^T, b\}$ is controllable

So, $\{c, A - bk^T\}$ is observable iff $\{A - bk^T, b, c\}$ is minimal.

$\{A - bk^T, b, c\}$ is minimal iff

$$c^T(sI - A + bk^T)^{-1}b = \frac{c^T Adj(sI - A + bk^T) b}{det(sI - A + bk^T)} = \frac{b(s)}{a_k(s)} \text{ is irreducible}$$

where $b(s) = c^T Adj(sI - A) b$ and $a_k(s) = det(sI - A + bk^T)$.

Hence, $\{c, A - bk^T\}$ is observable iff $a_k(s)$ and $b(s)$ have no common factors.

#2. $\dot{x} = Ax + b(v - ly)$, $y = c^Tx$ $\Rightarrow$ $\dot{x} = (A - lbc^T)x + bv$

$$\Rightarrow \underbrace{\begin{bmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix}}_{\mathcal{Y}(t)} = \frac{1}{1+l}\left\{ \underbrace{\begin{bmatrix} c^T \\ c^TA \\ \vdots \\ c^TA^{n-1} \end{bmatrix}}_{\mathcal{O}(c,A)} x(t) + \underbrace{\begin{bmatrix} 0 & 0 & & 0 \\ c^Tb & 0 & & 0 \\ c^TAb & c^Tb & \cdots & 0 \\ \vdots & & & \vdots \\ c^TA^{n-2}b & c^TA^{n-3}b & & c^Tb \end{bmatrix}}_{\mathbb{I}} \underbrace{\begin{bmatrix} v(t) \\ \dot{v}(t) \\ \vdots \\ v^{(n-2)}(t) \end{bmatrix}}_{\mathcal{V}(t)} \right\}$$

$\{c, A\}$ observable $\rightarrow$ $x(t) = \mathcal{O}(c, A)^{-1}\{(1+l)\mathcal{Y}(t) - \mathbb{I}\mathcal{V}(t)\}$

$\rightarrow$ $\{c, A - lbc^T\}$ is observable for any l.

\# 3. For non-minimal system $\{A, b, c\}$, transform to general decomposition form $\{\bar{A}, \bar{b}, \bar{c}\}$, where:

$$\bar{A} = T^{-1}AT = \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix}, \quad \bar{b} = T^{-1}b = \begin{bmatrix} b_{co} \\ b_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix}, \quad \bar{c}^T = c^T T = \begin{bmatrix} C_{co} \\ 0 \\ C_{\bar{c}o} \\ 0 \end{bmatrix}^T$$

$$\Rightarrow \quad \dot{x} = Ax + b(v - k^T x) = (T\bar{A}T^{-1} - bk^T)x + bv$$

$$\Rightarrow T^{-1}\dot{x} = (\bar{A} - T^{-1}bk^T T)\, T^{-1}x + T^{-1}bv$$

$$z = T^{-1}x \;\Rightarrow\; \dot{z} = (\bar{A} - \bar{b}\bar{k}^T)z + \bar{b}v, \quad y = \bar{c}^T z, \quad \bar{k} = \begin{bmatrix} k_{co} \\ k_{c\bar{o}} \\ k_{\bar{c}o} \\ k_{\bar{c}\bar{o}} \end{bmatrix}$$

$$\bar{A} - \bar{b}\bar{k}^T = \begin{bmatrix} A_{co} - b_{co}k_{co}^T & -b_{co}k_{c\bar{o}}^T & A_{13} - b_{co}k_{\bar{c}o}^T & -b_{co}k_{\bar{c}\bar{o}}^T \\ A_{21} - b_{c\bar{o}}k_{co}^T & A_{c\bar{o}} - b_{c\bar{o}}k_{c\bar{o}}^T & A_{23} - b_{c\bar{o}}k_{\bar{c}o}^T & A_{24} - b_{c\bar{o}}k_{\bar{c}\bar{o}}^T \\ 0 & 0 & A_{\bar{c}o} & 0 \\ 0 & 0 & A_{43} & A_{\bar{c}\bar{o}} \end{bmatrix}$$

$$\Rightarrow H_k(s) \triangleq c^T(sI - A + bk^T)^{-1}b = \bar{c}^T(sI - \bar{A} + \bar{b}\bar{k}^T)^{-1}\bar{b}$$

$$= C_{co}^T(sI - A_{co} + b_{co}k_{co}^T)^{-1}b_{co} = \frac{C_{co}^T \operatorname{Adj}(sI - A_{co} + b_{co}k_{co}^T)\, b_{co}}{\det(sI - A_{co} + b_{co}k_{co}^T)}$$

$$= \frac{C_{co}^T \operatorname{Adj}(sI - A_{co})\, b_{co}}{a_k(s)}, \quad a_k(s) = \det(sI - A_{co} + b_{co}k_{co}^T)$$

part #1 $\Rightarrow$ minimal subsystem $\{A_{co}, b_{co}, C_{co}\}$ will stay observable with linear state feedback iff $a_k(s)$ and $b(s)$ are relatively prime (where $b(s) = c^T \operatorname{Adj}(sI - A)\, b = C_{co}^T \operatorname{Adj}(sI - A_{co})\, b_{co}$)

Part #2 $\Rightarrow$ minimal part will stay observable always for case of linear output feedback