E6201 Linear Systems

HW9 – Solution

1) $\{A, b, c\}$, $\operatorname{rank}(\mathcal{O}(c,A)) = r < n$

$c,\ A^T c,\ \cdots\ (A^T)^{r-1} c$ are lin. ind

$\Rightarrow \exists$ $n \times n$ matrix E s.t.

$$\mathcal{O}(c,A)\,E = \begin{bmatrix} I & 0 \\ X & 0 \end{bmatrix}$$

Let $\bar{A} = E^{-1} A E$, $\bar{c}^T = c^T E$, $\bar{b} = E^{-1} b$.

Then $\mathcal{O}(\bar{c}, \bar{A}) = \mathcal{O}(c, A) E$ and $c^T (sI-A)^{-1} b = \bar{c}^T (sI - \bar{A})^{-1} \bar{b}$

$$\bar{A} \triangleq \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \\ \bar{A}_3 & \bar{A}_4 \end{bmatrix}, \quad \bar{c} \triangleq \begin{bmatrix} \bar{c}_1 \\ \bar{c}_2 \end{bmatrix}; \quad \bar{b} \triangleq \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \bar{c}^T = [e_1^T \ 0] \Rightarrow \bar{c}_2 = 0 \\ \mathcal{O}(\bar{c}_1, \bar{A}_1) = I \quad (\text{i.e. } \{\bar{c}_1, \bar{A}_1\} \text{ is} \\ \mathcal{O}(\bar{c}_1, \bar{A}_1)\bar{A}_2 = 0 \Rightarrow \bar{A}_2 = 0. \end{cases}$$

Hence,

$$c^T (sI-A)^{-1} b = [\bar{c}_1^T \ 0] \begin{bmatrix} sI - \bar{A}_1 & 0 \\ -\bar{A}_3 & sI - \bar{A}_4 \end{bmatrix}^{-1} \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{bmatrix}$$

$$= [\bar{c}_1^T \ 0] \begin{bmatrix} (sI - \bar{A}_1)^{-1} & 0 \\ X & (sI - \bar{A}_4)^{-1} \end{bmatrix} \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{bmatrix} = \bar{c}_1^T (sI - \bar{A}_1)^{-1} \bar{b}_1.$$

2) $\{A, b, c\}$ minimal realization of $H(s) = c^T (sI - A)^{-1} b$.

$c^T (sI-A)^{-1} b = b^T (sI - A^T)^{-1} c \Rightarrow \{A^T, c, b\}$ is also minimal realization

$\Rightarrow \exists\ T$ s.t. $A^T = T^{-1} A T$, $c = T^{-1} b$, $b^T = c^T T$.

$\Rightarrow A^T = (T^T)^{-1} A T^T \Rightarrow \mathcal{O}(c, A) = \mathcal{O}(T^{-1} b, T^T A^T (T^T)^{-1}) = \mathcal{O}((T^{-1})^T b, T A^T T^{-1})$

$\Rightarrow \mathcal{O}(b, A^T)(T^T)^{-1} = \mathcal{O}(b, A^T) T^{-1} \Rightarrow T = T^T.$

3) $\{A, b, c\}$ minimal

$a(s) = \det(sI - A)$ has repeated root λ

Assume A is similar to diagonal matrix

$\Rightarrow \exists\, t_1, t_2 \in \mathbb{C}^n$ s.t. $t_1^T A = \lambda t_1^T$, $t_2^T A = \lambda t_2^T$, t_1, t_2 lin. ind.

$\Rightarrow \exists\, \alpha_1, \alpha_2 \in \mathbb{C}$ s.t $(\alpha_1 t_1 + \alpha_2 t_2)^T b = 0$, $\alpha_1 \neq 0$, $\alpha_2 \neq 0$

$\Rightarrow (\alpha_1 t_1 + \alpha_2 t_2)^T A^k b = \lambda^k (\alpha_1 t_1 + \alpha_2 t_2)^T b = 0.$

$\Rightarrow (\alpha_1 t_1 + \alpha_2 t_2)^T \mathcal{C}(A, b) = 0.$ contradicts $\{A, b, c\}$ is minimal

$\Rightarrow$ A is not similar to diagonal matrix