E6201 Linear Systems

HW#8 Solutions

1] Let $A = T J_A T^{-1}$, $T = [t_1^{(1)} \cdots t_{l_p}^{(p)}]$. S is inv. subspace of A.

$S \neq 0 \Rightarrow \exists\ k \in 1, \dots, p$ and $q \in 1, \dots, l_k$ s.t. $t_q^{(k)} \in S$.

Then, $A t_q^{(k)} = \lambda_k t_q^{(k)} + t_{q-1}^{(k)} \in S \Rightarrow t_{q-1}^{(k)} \in S$

similarly, $t_{q-2}^{(k)} \in S, \dots, t_1^{(k)} \in S \Rightarrow S$ contains ch. vector of A

2] $A = T J_A T^{-1}$, $T = [t_1^{(1)}, \dots, t_{l_p}^{(p)}]$, $S_k = \text{span}\{t_1^{(k)}, \dots, t_{l_k}^{(k)}\}$

$y \in S_k$, $y \neq 0 \Rightarrow y = \alpha_1 t_1^{(k)} + \dots + \alpha_{l_k} t_{l_k}^{(k)}$.

$Ay = \alpha_1 \lambda_k t_1^{(k)} + \alpha_2 (\lambda_k t_2^{(k)} + t_1^{(k)}) + \dots + \alpha_{l_k} (\lambda_k t_{l_k}^{(k)} + t_{l_k-1}^{(k)})$

$\Rightarrow Ay \in S_k \Rightarrow S_k$ is inv. subspace of A

3] Let $X(t) = \begin{bmatrix} \xi_0(t) \\ \vdots \\ \xi_{n-1}(t) \end{bmatrix}$. Then $\frac{d}{dt} X(t) = A_{co} X(t)$, $A_{co} = \begin{bmatrix} 0 & \cdots & 0 & -a_n \\ 1 & & 0 & -a_{n-1} \\ & \ddots & & \\ 0 & & 1 & -a_1 \end{bmatrix}$

$\Rightarrow X^{(n)}(t) = A_{co} X^{(n-1)}(t) = \dots = A_{co}^{n-1} X(t)$

$e^{At}|_{t=0} \Rightarrow X(0) = [1\ 0 \cdots 0]^T$

suppose $c^T X(t) = 0$, $-\infty < t < \infty$.

$\Rightarrow c^T X(0) = 0 \Rightarrow c^T X^{(1)}(0) = A_{co} X(0) = 0 \Rightarrow$

$\dots \Rightarrow c^T X^{(n-1)}(0) = c^T A_{co}^{n-1} X(0) = 0$

$$\left.\begin{array}{l} 0 = c^T X(0) \Rightarrow c_1 = 0 \\ 0 = c^T A X(0) \Rightarrow c_2 = 0 \\ \vdots \\ 0 = c^T A^{n-1} X(0) \Rightarrow c_n = 0 \end{array}\right\} \Rightarrow \xi_0(t), \dots, \xi_{n-1}(t) \text{ are lin. ind. on any interval}$$

4] a] $A = A^* \Rightarrow A^*A = AA^* \Rightarrow A = U\Lambda U^*,\ U^{-1} = U^*,\ \Lambda = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}$

$$\Rightarrow U\Lambda U^* = U\Lambda^* U^* \Rightarrow \Lambda = \Lambda^*$$

converse: $A = U\Lambda U^*,\ U^{-1} = U^*,\ \Lambda = \Lambda^*$

$$\Rightarrow A^* = U\Lambda^* U^* = U\Lambda U^* = A \Rightarrow A \text{ self-adjoint}$$

b] $A = A^T = A^* \Rightarrow A = U\Lambda U^*,\ U^{-1} = U^*,\ \Lambda = \Lambda^*$

Suppose $u_1 = a + jb$ s.t. $a \in \mathbb{R}^n, b \in \mathbb{R}^n, a \neq 0, b \neq 0$, and $Au_1 = \lambda_1 u_1$

Let $u_2 = \bar{u}_1$. Then $Au_2 = \lambda u_2$, where $u_2^* u_1 = 0$.

$$\Rightarrow u_1^T u_1 = 0 \Rightarrow a^T a = b^T b,\ a^T b = 0, \text{ where } Aa = \lambda_1 a \text{ and } Ab = \lambda_1 b.$$

I.e. can replace complex u_1 and u_2 with real $q_1 \triangleq \frac{a}{\|a\|}$, $q_2 \triangleq \frac{b}{\|b\|}$

s.t. $A = Q\Lambda Q^T,\ Q = [q_1 \cdots q_n],\ Q^{-1} = Q^* = Q^T$.

(if u_j is pure real then $q_j \triangleq u_j$ and if u_j is pure imaginary then $q_j \triangleq i q_j$.)

converse: $A = Q\Lambda Q^T,\ \Lambda = \Lambda^*,\ Q^{-1} = Q^* = Q^T$

$$A^T = Q\Lambda Q^T = A \quad \text{and} \quad A^* = \bar{Q}\Lambda^* Q^* = Q\Lambda Q^T = A.$$

5] $X(s) = \int_0^\infty x(t) e^{-st}\, dt,\ \mathrm{Re}(s) > -a,\ a > 0 \Rightarrow |X(j\omega)| < \infty \quad \forall \omega.$

$$\Rightarrow x(t) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} X(s) e^{st}\, ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$$

$$\Rightarrow \overline{x(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{X(j\omega)}\, e^{-j\omega t}\, d\omega$$

$$2\pi\, \delta(\omega_2 - \omega_1) = \int_0^\infty e^{-j(\omega_2 - \omega_1)t}\, dt$$

$$\Rightarrow \int_0^\infty |x(t)|^2\, dt = \int_0^\infty \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega_1) e^{j\omega_1 t}\, d\omega_1 \right)\left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{X(j\omega_2)}\, e^{-j\omega_2 t}\, d\omega_2 \right) dt$$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(j\omega_1) \overline{X(j\omega_2)} \int_0^\infty e^{-j(\omega_2 - \omega_1)t}\, dt\, d\omega_1\, d\omega_2$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X(j\omega_1) \overline{X(j\omega_2)}\, \delta(\omega_2 - \omega_1)\, d\omega_1\, d\omega_2$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2\, d\omega.$$