E6201 Linear Systems
Homework 8 (*due:* Mar. 29)

1. Suppose $\mathcal{S} \triangleq \mathrm{span}(t_1, \dots, t_l)$ is an invariant subspace of A, i.e. if $x \in \mathcal{S}$, then $Ax \in \mathcal{S}$. Show there exists a characteristic vector of A in $\mathcal{S}$.

2. Let $A = TJ_AT^{-1}$, where

$$J_A = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_p \end{bmatrix},$$

J_k denotes an $l_k \times l_k$ Jordan block with characteristic value λ_k and $\left\{t_1^{(k)}, t_2^{(k)}, \dots, t_{l_k}^{(k)}\right\}$ denote a Jordan chain of vectors of corresponding to J_k. In particular,

$$(A - \lambda_k I)t_1^{(k)} = 0, \qquad (A - \lambda_k I)t_q^{(k)} = t_{q-1}^{(k)}, \qquad q = 2, \dots, l_k, \qquad k = 1, \dots, p,$$

and $T = \left[t_1^{(1)},\ t_2^{(1)}, \dots, t_{l_p}^{(p)}\right]$.

Show $\mathcal{S}_k = \mathrm{span}\left(t_1^{(k)}, \dots, t_{l_k}^{(k)}\right)$ is an invariant subspace of A.

3. Show there exists n linearly independent functions $\xi_0(t), \xi_1(t), \dots, \xi_{n-1}(t)$, $-\infty < t < \infty$, such that

$$e^{At} = \xi_0(t)I + \xi_1(t)A + \cdots + \xi_{n-1}(t)A^{n-1}.$$

[*Hint:* See Kailath Sec. 2.5, p. 167.]

4. In class we showed $A^*A = AA^*$ if and only if $A = U\Lambda U^*$, where $U^{-1} = U^*$,

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix},$$

and $\lambda_1, \dots, \lambda_n$ denote the characteristic values of A.

(a) Show $A = A^*$ if and only if $A = U\Lambda U^*$, where $\Lambda = \Lambda^*$ and $U^{-1} = U^*$.

(b) Show $A = A^* = A^T$ if and only if $A = Q\Lambda Q^T$, where $\Lambda = \Lambda^*$ and $Q^{-1} = Q^* = Q^T$.

5. Show Parseval's theorem:

$$\int_0^\infty |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^\infty |X(j\omega)|^2\, d\omega,$$

where $X(s) = \int_0^\infty x(t)e^{-st}\, dt$, $\mathrm{Re}(s) > -a,\ a > 0$.