E6201 Linear Systems

HW #7 solutions

1] $$\frac{d}{dt} e^{At} = \frac{d}{dt} \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = \sum_{k=1}^{\infty} \frac{A^k t^{k-1}}{(k-1)!} = A \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = A e^{At}$$

$$\Rightarrow \frac{d^n}{dt^n} e^{At} = A^n e^{At}$$

2] 2x2 case: $$C_c^{(2)} \cdot [\underline{a}^{(2)}]^T = \begin{bmatrix} 1 & -a_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

general case: assume $[C_c^{(n-1)}]^{-1} = [\underline{a}^{(n-1)}]^T$

$$A_c^{(n)} = \begin{bmatrix} A_c^{(n-1)} & \begin{bmatrix} -a_n \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ [0 \cdots 0\ 1] & 0 \end{bmatrix}$$

$$b^{(n)} = [[b^{(n-1)}]^T\ 0]^T$$

$$[C_c^{(n)}]^{-1} = \begin{bmatrix} C_c^{(n-1)} & [A_c^{(n-1)}]^{n-1} b^{(n-1)} \\ [0 \cdots 0] & 1 \end{bmatrix}^{-1}$$

$$\overset{(A.22)}{=} \begin{bmatrix} [\underline{a}^{(n-1)}]^T & \begin{bmatrix} a_{n-1} \\ \vdots \\ a_1 \end{bmatrix} \\ [0 \cdots 0] & 1 \end{bmatrix} = [\underline{a}^{(n)}]^T$$

[where, $[A_c^{(n-1)}]^{n-1} \underline{b}^{(n-1)} = \sum_{j=1}^{n-1} -a_j [A_c^{(n-1)}]^{n-j-1} \underline{b}^{(n-1)} = C_c^{(n-1)}([-a_{n-1} \cdots -a_1]^T)$.

3] Suppose $\dot{x} = Ax + bu$, $y = c^T x$

and let $z = T^{-1} x$. Then:

$$\mathcal{C}[T^{-1}AT, T^{-1}b] = [T^{-1}b \quad T^{-1}Ab \cdots T^{-1}A^{n-1}b] = T^{-1} \mathcal{C}[A, b]$$

$\Rightarrow$ z is controllable iff x is controllable

and $$\mathcal{O}[T^T c, T^{-1}AT] = \begin{bmatrix} c^T T \\ c^T A T \\ \vdots \\ c^T A^{n-1} T \end{bmatrix} = \mathcal{O}[c, A] \cdot T$$

$\Rightarrow$ z is observable iff x is observable.

2.3-12 From Exercise 2.2-22:

$$\dot{x} = \begin{bmatrix} -2R/L & 1/L \\ -1/C & 0 \end{bmatrix} x + \begin{bmatrix} R/L \\ 1/C \end{bmatrix} u \,, \qquad y = [-R \;\; 1]\, x + Ru$$

$$\mathcal{C} = [\, b \;\; Ab \,] = \begin{bmatrix} R/L & -2R^2/L + 1/LC \\ 1/C & -R/LC \end{bmatrix}$$

$$\det(\mathcal{C}) = -\frac{R^2}{L^2C} - \frac{1}{C}\left(\frac{1}{LC} - \frac{2R^2}{L}\right) = \frac{-R^2C - L + 2R^2LC}{L^2C^2} = 0$$

$$\Rightarrow \quad -R^2C - L + 2R^2LC = 0$$

$$CR^2(2L-1) = L$$

$\Rightarrow$ if $CR^2 = \frac{L}{2L-1}$ then circuit is uncontrollable

$$\mathcal{O} = \begin{bmatrix} c^T \\ c^TA \end{bmatrix} = \begin{bmatrix} -R & 1 \\ \frac{2R^2}{L} - \frac{1}{C} & -R/L \end{bmatrix}$$

$\det(\mathcal{O}) = \frac{R^2}{L} - \frac{2R^2}{L} + \frac{1}{C} = 0 \Rightarrow$ if $R^2C = L$ then circuit is unobservable

From Exercise 2.2-23: (solution 1 assumes resistor is excluded)

$$\dot{x} = \begin{bmatrix} 0 & 0 & L^{-1} & 0 \\ 0 & 0 & 0 & L^{-1} \\ -C^{-1} & 0 & 0 & 0 \\ 0 & -C^{-1} & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ C^{-1} \\ C^{-1} \end{bmatrix} u \qquad y = [0 \; 0 \; 1 \; 1]\, x$$

$$\mathcal{C} = [\, b \;\; Ab \;\; A^2b \;\; A^3b \,] = \begin{bmatrix} 0 & (LC)^{-1} & 0 & -L^{-2}C^{-2} \\ 0 & (LC)^{-1} & 0 & -L^{-2}C^{-2} \\ C^{-1} & 0 & -L^{-1}C^{-2} & 0 \\ C^{-1} & 0 & -L^{-1}C^{-2} & 0 \end{bmatrix}$$

$\Rightarrow$ uncontrollable $\forall\, R, L, C$

$$\mathcal{O} = \begin{bmatrix} c^T \\ c^TA \\ c^TA^2 \\ c^TA^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ -C^{-1} & -C^{-1} & 0 & 0 \\ 0 & 0 & -L^{-1}C^{-1} & -L^{-1}C^{-1} \\ L^{-1}C^{-2} & L^{-1}C^{-2} & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ unobservable. $\forall\, R, L, C$

(solution 2 assumes resistor is included)

$$A = \begin{bmatrix} -L^{-1}R & -L^{-1}R & L^{-1} & 0 \\ -L^{-1}R & -L^{-1}R & 0 & L^{-1} \\ -C^{-1} & 0 & 0 & 0 \\ 0 & -C^{-1} & 0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} L^{-1}R \\ L^{-1}R \\ C^{-1} \\ C^{-1} \end{bmatrix} \quad c = \begin{bmatrix} -R \\ -R \\ 1 \\ 1 \end{bmatrix}$$

$\det(\mathcal{C}(A,b)) = 0 \Rightarrow$ uncontrollable $\forall\, R, L, C$

$\det(\mathcal{O}(c,A)) = 0 \Rightarrow$ unobserbable $\forall\, R, L, C$

2.3-14 Show that $\left\{ \begin{bmatrix} A & 0 \\ c^T & 0 \end{bmatrix}, \begin{bmatrix} b \\ 0 \end{bmatrix} \right\}$ is controllable iff $\det(\mathcal{C}(A,b)) \neq 0$ and $\begin{bmatrix} A & b \\ c^T & 0 \end{bmatrix}$ has full rank.

(implies that the output of the system $\{A,b,c\}$ can be set to an arbitrary value)

pf:

$$\mathcal{C}\left(\underbrace{\begin{bmatrix} A & 0 \\ c^T & 0 \end{bmatrix}}_{\tilde{A}}, \underbrace{\begin{bmatrix} b \\ 0 \end{bmatrix}}_{\tilde{b}}\right) = \begin{bmatrix} b & Ab & A^2b & \cdots & A^nb \\ 0 & c^Tb & c^TAb & & c^TA^{n-1}b \end{bmatrix}$$

$$= \left[\begin{matrix} b \\ 0 \end{matrix} \; \begin{bmatrix} A \\ c^T \end{bmatrix} [b \; Ab \cdots A^{n-1}b] \right] = \left[\begin{matrix} b \\ 0 \end{matrix} \; \begin{bmatrix} A \\ c^T \end{bmatrix} \mathcal{C}(A,b) \right]$$

$$\triangleq \mathcal{C}(\tilde{A}, \tilde{b})$$

(#1): $\begin{bmatrix} A \\ c^T \end{bmatrix} \mathcal{C}(A,b)$ is rank-n.

pf: $\det\left(\begin{bmatrix} A & b \\ c^T & 0 \end{bmatrix}\right) \neq 0 \iff \det(A) \neq 0$ (Appendix A.22)

$\iff A\,\mathcal{C}(A,b)$ is rank-n (since $\mathcal{C}(A,b)$ is rank n)

$\iff \begin{bmatrix} A \\ c^T \end{bmatrix} \mathcal{C}(A,b)$ is rank-n

(#2): $\left[\begin{matrix} b \\ 0 \end{matrix} \; \begin{bmatrix} A \\ c^T \end{bmatrix} \mathcal{C}(A,b) \right]$ is rank-$(n+1)$

pf: suppose that rank of $\left[\begin{matrix} b \\ 0 \end{matrix} \; \begin{bmatrix} A \\ c^T \end{bmatrix} \mathcal{C}(A,b) \right]$ is n, then by (#1) $\exists\, \underline{\alpha}$ s.t. $\begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} A \\ c^T \end{bmatrix} \mathcal{C}(A,b)\,\underline{\alpha} \Rightarrow \begin{bmatrix} b \\ 0 \end{bmatrix} = \begin{bmatrix} A \\ c^T \end{bmatrix} \underline{\beta}$, $\underline{\beta} = \mathcal{C}(A,b)\,\underline{\alpha}$,

contradicts assumption that $\det\left(\begin{bmatrix} A & b \\ c^T & 0 \end{bmatrix}\right) \neq 0$.

$\Rightarrow \det(\mathcal{C}[\tilde{A}, \tilde{b}]) \neq 0$ iff $\det\left(\begin{bmatrix} A & b \\ c^T & 0 \end{bmatrix}\right) \neq 0$ and $\det(\mathcal{C}[A,b]) \neq 0$.