E6201 HW6 – Solutions

1]
$$\lim_{t\to\infty}\frac{t^n}{e^{st}} = \lim_{t\to\infty}\frac{nt^{n-1}}{se^{st}} = \cdots = \lim_{t\to\infty}\frac{n!}{s^n e^{st}} = 0.$$

$$\Rightarrow \int_0^\infty t^n e^{-st}\,dt = -\frac{t^n}{s}e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty n t^{n-1} e^{-st}\,dt$$

$$= n s^{-1}\int_0^\infty t^{n-1} e^{-st}\,dt = \cdots = n!\, s^{-n-1}$$

by assumption, $\exists\ \varepsilon > 0$ s.t.

$$x(t) = \sum_{n=0}^{\infty}\frac{x^{(n)}(0)}{n!}t^n, \quad t\in(-\varepsilon,\varepsilon)$$

$$\Rightarrow X(s) = \sum_{n=0}^{\infty}\frac{x^{(n)}(0)}{s^{n+1}}$$

2]
$$y^{(2)}(0) = y^{(1)}(0) = y(0) = u^{(1)}(0) = u(0) \Rightarrow$$

$$(s^3 + a_1 s^2 + a_2 s + a_3)\,Y(s) = (b_1 s^2 + b_2 s + b_3)\,U(s)$$

$$\Rightarrow H(s) = \frac{Y(s)}{U(s)} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

3]
$$X(s) = (sI-A)^{-1}[bU(s) + X(0)], \quad Y(s) = c^T X(s)$$

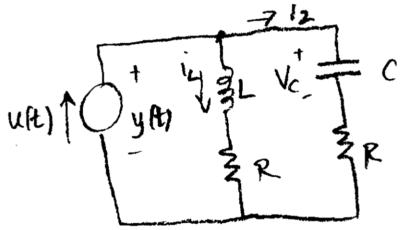
$$\Rightarrow H(s) = c^T(sI-A)^{-1}b = \frac{c^T\,\mathrm{Adj}(sI-A)\,b}{\det(sI-A)}$$

4]
$$(sI-A_c)^{-1}b_c = \frac{[s^{n-1}\ s^{n-2}\ \cdots\ 1]^T}{a(s)}$$

$$\Rightarrow H(s) = \frac{c_c^T(sI-A)^{-1}b_c}{a(s)} = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

5] 2.2-22 and 2.2-23 attached.

2.2-22



$$X_1(t) = i_L(t)$$
$$X_2(t) = V_c(t) = C^{-1}\int_0^t i_2(t)\,dt$$
$$\dot{X}_2 = C^{-1} i_2 = C^{-1}(u - X_1)$$

a) $V_L = L\frac{di_L}{dt} = L\dot{X}_1$; $u(t) = i_L + i_2 = X_1 + i_2$

$$i_L R + V_L - V_c - i_2 R = 0 \qquad i_2 = u - X_1$$

$$R X_1 + L\dot{X}_1 - X_2 - (u - X_1)R = 0$$

$$\dot{X}_1 = \frac{-2RX_1 + X_2 + Ru}{L} \Rightarrow \begin{bmatrix}\dot{X}_1\\ \dot{X}_2\end{bmatrix} = \begin{bmatrix}-\frac{2R}{L} & 1/L\\ -C^{-1} & 0\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix} + \begin{bmatrix}R/L\\ C^{-1}\end{bmatrix}u$$

$$y = X_2 + (u - X_1)R = \begin{bmatrix}-R & 1\end{bmatrix}\begin{bmatrix}X_1\\ X_2\end{bmatrix} + Ru$$

b) $H(s) = c^T(sI - A)^{-1} b + d$

$$(sI-A)^{-1} = \begin{bmatrix}s + \frac{2R}{L} & -1/L\\ \frac{1}{C} & s\end{bmatrix}^{-1} = \frac{1}{s(s+\frac{2R}{L}) + \frac{1}{LC}}\begin{bmatrix}s & 1/L\\ -\frac{1}{C} & s + \frac{2R}{L}\end{bmatrix}$$

$$c^T(sI-A)^{-1}b = \frac{[-R \quad 1]}{s(s+\frac{2R}{L}) + \frac{1}{LC}}\begin{bmatrix}sR/L + 1/LC\\ -\frac{R}{LC} + \frac{s}{C} + \frac{2R}{LC}\end{bmatrix} = \frac{-\frac{sR^2}{L} - \cancel{\frac{R}{LC}} - \frac{R}{LC} + \frac{s}{C} + \cancel{\frac{2R}{LC}}}{s(s+\frac{2R}{L}) + \frac{1}{LC}}$$

$$\Rightarrow H(s) = \frac{R(s^2 + \frac{2Rs}{L} + \frac{1}{LC}) - \frac{sR^2}{L} + \frac{s}{C}}{s^2 + \frac{2Rs}{L} + \frac{1}{LC}}$$

$$= \frac{Rs^2 + \frac{R^2 s}{L} + \frac{s}{C} + \frac{R}{LC}}{s^2 + \frac{2Rs}{L} + \frac{1}{LC}}$$

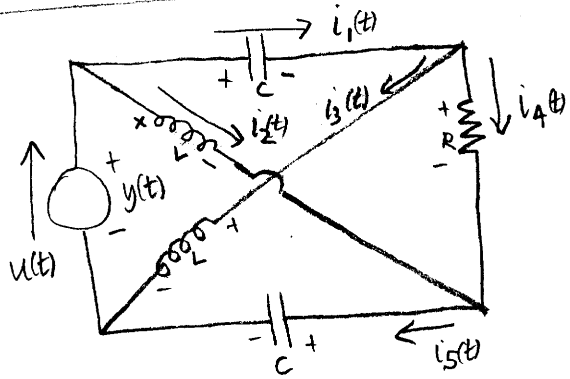
Note: if $R^2 = L/C$, then $H(s) = \dfrac{R\left(Rs^2 + \frac{2s}{C} + \frac{R}{LC}\right)}{Rs^2 + \frac{2s}{C} + \frac{R}{LC}} = R$

$$\Rightarrow Y(s) = R\,U(s)$$

$$\Rightarrow y(t) = R\,u(t)$$ "constant resistance network"

Linear Systems Spring 2012

Update to HW#6 solution 5b. [2.2-23]



$$V_{C_1}(t) = C^{-1}\int_0^t i_1(t)\,dt$$
$$V_{C_5}(t) = C^{-1}\int_0^t i_5(t)\,dt$$
$$V_{L_2}(t) = L\frac{di_2(t)}{dt}$$
$$V_{L_3}(t) = L\frac{di_3(t)}{dt}$$
$$V_{R_4}(t) = R\,i_4(t)$$

(1.) $u = i_1 + i_2$

(2.) $u = i_3 + i_5$

(3.) $y = V_{C_1} + V_{R_4} + V_{C_5}$

(4.) $V_{L_3} = V_{R_4} + V_{C_5}$

(5.) $V_{L_2} = V_{R_4} + V_{C_1}$

(6.) $i_1 = i_3 + i_4$

$x_1 = i_2 \Rightarrow \dot{x}_1 = L^{-1}V_{L_2}$

$x_2 = i_3 \Rightarrow \dot{x}_2 = L^{-1}V_{L_3}$

$x_3 = V_{C_1} \Rightarrow \dot{x}_3 = C^{-1}i_1$

$x_4 = V_{C_5} \Rightarrow \dot{x}_4 = C^{-1}i_5$

(1.) $\Rightarrow \dot{x}_3 = C^{-1}u - C^{-1}x_1$

(2.) $\Rightarrow \dot{x}_4 = C^{-1}u - C^{-1}x_2$

(6.) $\Rightarrow V_{R_4} = R(i_1 - i_3) = (C\dot{x}_3 - x_2)R$
$= Ru - Rx_1 - Rx_2$

(3.) $\Rightarrow y = x_3 + Ru - Rx_1 - Rx_2 + x_4$

(4.) $\Rightarrow \dot{x}_2 = L^{-1}(x_4 + Ru - Rx_1 - Rx_2)$

(5.) $\Rightarrow \dot{x}_1 = L^{-1}(x_3 + Ru - Rx_1 - Rx_2)$

$$\dot{\underline{x}} = \begin{bmatrix} -L^{-1}R & -L^{-1}R & L^{-1} & 0 \\ -L^{-1}R & -L^{-1}R & 0 & L^{-1} \\ -C^{-1} & 0 & 0 & 0 \\ 0 & -C^{-1} & 0 & 0 \end{bmatrix}\underline{x} + \begin{bmatrix} L^{-1}R \\ L^{-1}R \\ C^{-1} \\ C^{-1} \end{bmatrix}u, \qquad y = [-R\ -R\ 1\ 1]\underline{x} + Ru$$

$$H(s) = [-R\ -R\ 1\ 1]\begin{bmatrix} s+L^{-1}R & L^{-1}R & -L^{-1} & 0 \\ L^{-1}R & s+L^{-1}R & 0 & -L^{-1} \\ C^{-1} & 0 & s & 0 \\ 0 & C^{-1} & 0 & s \end{bmatrix}^{-1}\begin{bmatrix} L^{-1}R \\ L^{-1}R \\ C^{-1} \\ C^{-1} \end{bmatrix} + R$$

$$= \frac{2(L - CR^2)s}{1 + Cs(2R + Ls)} + R$$

$$L = CR^2 \Rightarrow H(s) = R$$