EG201 Linear Systems

HW#5 — solutions

1| $\mathcal{L}[\dot{x}(t)] = s\mathcal{L}[x(t)] - x(0) \Rightarrow$

$$\mathcal{L}[\ddot{x}(t)] = s\mathcal{L}[\dot{x}(t)] - \dot{x}(0) = s^2\mathcal{L}[x(t)] - sx(0) - \dot{x}(0)$$

2| $\int p_\Delta(t)\, f(t)\, dt = f(t_\Delta), \quad t_\Delta \in [0, \Delta)$

$$\Rightarrow \lim_{\Delta\to 0} \int p_\Delta(t) f(t)\, dt = f(0).$$

3| Let $H^{(n)} = \begin{bmatrix} \lambda^{-1} & -\lambda^{-2} & \lambda^{-3} & -\lambda^{-4} & \cdots & (-1)^{n-1}\lambda^{-n} \\ 0 & \lambda^{-1} & -\lambda^{-2} & \lambda^{-3} & \cdots & \vdots \\ 0 & 0 & \lambda^{-1} & -\lambda^{-2} & & \vdots \\ & & & \ddots & & -\lambda^{-2} \\ & & & & & \lambda^{-1} \end{bmatrix}$ where $\lambda \neq 0$,

and let $J^{(n)} = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda \end{bmatrix} \Rightarrow J^{(2)}H^{(2)} = I \Rightarrow H^{(2)} = [J^{(2)}]^{-1}$

Assume $H^{(n-1)} = [J^{(n-1)}]^{-1}$, then by inverse of block matrix (A.22)

$$[J^{(n)}]^{-1} = \begin{bmatrix} J^{(n-1)} & \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \\ [0 \cdots 0] & \lambda \end{bmatrix}^{-1} = \begin{bmatrix} H^{(n-1)} & -H^{(n-1)}\begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}\lambda^{-1} \\ [0 \cdots 0] & \lambda^{-1} \end{bmatrix}$$

$= H^{(n)}$. Hence, $\lambda_1 \neq 0, \cdots, \lambda_p \neq 0 \Rightarrow \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix}^{-1} = \begin{bmatrix} J_1^{-1} & & \\ & \ddots & \\ & & J_p^{-1} \end{bmatrix}$

4| $a(\lambda) = 0 \Rightarrow \lambda^n = -a_1\lambda^{n-1} - \cdots - a_n$

$$\Rightarrow A_c \begin{bmatrix} \lambda^{n-1} \\ \lambda^{n-2} \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} -a_1\lambda^{n-1} - \cdots - a_n \\ \lambda^{n-1} \\ \vdots \\ \lambda \end{bmatrix} = \lambda \begin{bmatrix} \lambda^{n-1} \\ \vdots \\ 1 \end{bmatrix}.$$

5]

$$y(t) = \frac{1}{C}\int_0^t i(\tau)\,d\tau \qquad y(0) = 1$$

$$\dot y(t) = \frac{1}{C}\, i(t) \qquad i(0) = 0 \Rightarrow \dot y(0) = 0$$

$$\ddot y(t) = \frac{1}{C}\frac{d}{dt} i(t) \qquad R = 2,\ L = 1,\ C = \frac{1}{2}$$

$$\Rightarrow \quad u(t) = LC\,\ddot y(t) + RC\,\dot y(t) + y(t)$$

$$\Rightarrow \quad 2u(t) = \ddot y(t) + 2\dot y(t) + 2y(t)$$

$$\Rightarrow \quad 2U(s) = (s^2 + 2s + 2)Y(s) - (s+2)$$

$$= (s - \lambda)(s - \bar\lambda)\,Y(s) - (s+2),$$

where $\lambda = -(1+i)$

a] $u(t) = 0 \Rightarrow Y_0(s) = \dfrac{s+2}{(s-\lambda)(s-\bar\lambda)} = \dfrac{a}{s-\lambda} + \dfrac{\bar a}{s-\bar\lambda}$,

where $a = \dfrac{1+i}{2}$

$$\Rightarrow \quad y_0(t) = a e^{\lambda t} + \bar a e^{\bar\lambda t} = 2\,\mathrm{Re}\left(a e^{\lambda t}\right).$$

$$= e^{-t}\cos t + e^{-t}\sin t$$

b] $u(t) = \frac{1}{2}\exp(-at)$

$$\Rightarrow \quad Y(s) = Y_0(s) + \frac{U(s)}{(s-\lambda)(s-\bar\lambda)} = Y_0(s) + \left[\frac{(\lambda-\bar\lambda)^{-1}}{s-\lambda} + \frac{(\bar\lambda-\lambda)^{-1}}{s-\bar\lambda}\right] U(s)$$

$$\rightarrow \quad y(t) = Y_0(t) + \left[\frac{i}{2} e^{\lambda t} - \frac{i}{2} e^{\bar\lambda t}\right] * u(t)$$

$$= Y_0(t) + \frac{i}{4} e^{-at}\int_0^t \left(e^{\lambda\tau} - e^{\bar\lambda\tau}\right) e^{a\tau}\,d\tau$$

$$= Y_0(t) + \frac{i}{4} e^{-at}\left[\frac{1}{\lambda+a} e^{(\lambda+a)\tau} - \frac{1}{\bar\lambda+a} e^{(\bar\lambda+a)\tau}\right]_{\tau=0}^{t}$$

$$= Y_0(t) + \frac{i}{4} e^{-at}\left[\frac{1}{\lambda+a}\left(e^{(\lambda+a)t} - 1\right) - \frac{1}{\bar\lambda+a}\left(e^{(\bar\lambda+a)t} - 1\right)\right]$$

$$= Y_0(t) + \frac{1}{2}\,\mathrm{Re}\left[\frac{i}{\lambda+a}\left(e^{\lambda t} - e^{-at}\right)\right]$$

$$= Y_0(t) + \frac{1}{2(|\lambda|-a)^2}\left[e^{-at} + e^{-t}\left((a-1)\sin t - \cos t\right)\right].$$