EE201 Linear Systems — HW4 solutions, Spring 2012

1] $A \overset{E}{=} B$ denotes "A is equivalent to B". By elem. operations:

$$sI - J_k = \begin{bmatrix} s-\lambda_k & -1 & & \\ & s-\lambda_k & \ddots & \\ & & \ddots & -1 \\ & & & s-\lambda_k \end{bmatrix} \overset{E}{=} \begin{bmatrix} 1 & -1 & & \\ (s-\lambda_k-1)(s-\lambda_k) & s-\lambda_k & & \\ & & \ddots & -1 \\ & & & s-\lambda_k \end{bmatrix} \overset{E}{=} \begin{bmatrix} 1 & 0 & & \\ (s-\lambda_k-1)(s-\lambda_k) & (s-\lambda_k)^2 & & \\ & & \ddots & -1 \\ & & & s-\lambda_k \end{bmatrix}$$

$$\overset{E}{=} \begin{bmatrix} 1 & 0 & 0 & & \\ 0 & (s-\lambda_k)^2 & -1 & & \\ 0 & 0 & s-\lambda_k & \ddots & -1 \\ 0 & & & & s-\lambda_k \end{bmatrix} \overset{E}{=} \begin{bmatrix} 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ 0 & 0 & (s-\lambda_k)^3 & \ddots & -1 \\ & & & & s-\lambda_k \end{bmatrix} \overset{E}{=} \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & (s-\lambda_k)^{l_k} \end{bmatrix}$$

$$\Rightarrow \quad sI - J = \begin{bmatrix} sI-J_1 & & \\ & \ddots & \\ & & sI-J_p \end{bmatrix} \overset{E}{=} \begin{bmatrix} 1 \cdots (s-\lambda_1)^{l_1} & & \\ & 1 \cdots (s-\lambda_2)^{l_2} & \\ & & 1 \cdots (s-\lambda_p)^{l_p} \end{bmatrix} \triangleq C(s)$$

The invariant polynomials are given by:

$$h_1(s) = \delta_C^{(1)}(s), \quad h_k(s) = \frac{\delta_C^{(k)}(s)}{\delta_C^{(k-1)}(s)}, \quad k = 2, \dots, n,$$

where $\delta_C^{(k)}(s) = \text{g.c.m.d} \left\{ C\begin{pmatrix} i_1 \cdots i_k \\ j_1 \cdots j_k \end{pmatrix} : 1 \le i_1 \le \cdots \le i_k \le n, \ 1 \le j_1 \le \cdots \le j_k \le n \right\}$

$\lambda_j \neq \lambda_k$, $k \neq j$ $\Rightarrow$ $(s-\lambda_j)$ and $(s-\lambda_k)$ are relatively prime for $k \neq j$

$\Rightarrow \delta_C^{(k)}(s) = 1, \quad k = 1, \dots, n-1$ (since any polynomial divides zero without remainder)

and $\delta_C^{(n)}(s) = \prod_{k=1}^{p} (s-\lambda_k)^{l_k}$

$$\Rightarrow \quad C(s) \overset{E}{=} \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \prod(s-\lambda_k)^{l_k} \end{bmatrix}$$ (since matrices with the same invariant polynomials are equivalent)

$$\Rightarrow \quad sI - J \overset{E}{=} \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \prod(s-\lambda_k)^{l_k} \end{bmatrix}$$

2] Solution appears in lecture #4 notes p. 9.

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$h_j(s) = 1, \quad j = 1, \dots, k$

$h_j(s) = (s-\lambda_1)\cdots(s-\lambda_{p_j}), \quad j = k+1, \dots, n, \quad \lambda_j \neq \lambda_k$ for $j \neq k$

$$\Rightarrow \quad J_A = \begin{bmatrix} \lambda_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \lambda_{p_1} & & & & & \\ & & & \lambda_1 & & & & \\ & & & & \ddots & & & \\ & & & & & \lambda_{p_2} & & \\ & & & & & & \ddots & \\ & & & & & & & \lambda_{p_n} \end{bmatrix}$$

since each relatively prime factor of $h_j(s)$ has an exponent equal to one, $j = k+1, \dots, n$

i.e. J_A is diagonal.

Let T denote the similarity transformation matrix corresponding to J_A

i.e. $\quad A = T J_A T^{-1}$

Suppose $T = [t_1 \cdots t_n]$. Then $A t_k = \lambda_k t_k$, $\lambda_k \in \{\lambda_1, \dots, \lambda_{p_n}\}$.

Hence, A has n linearly independent ch. vectors.

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$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad sI - A = \begin{bmatrix} s-1 & 0 & 0 & 0 \\ -1 & s-1 & -2 & -3 \\ 0 & 0 & s-1 & 0 \\ 0 & 0 & 0 & s-1 \end{bmatrix}$$

$$sI - A \overset{E}{=} \begin{bmatrix} -1 & s-1 & -2 & -3 \\ s-1 & 0 & 0 & 0 \\ 0 & 0 & s-1 & 0 \\ 0 & 0 & 0 & s-1 \end{bmatrix} \overset{E}{=} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & (s-1)^2 & -2(s-1) & -3(s-1) \\ 0 & 0 & s-1 & 0 \\ 0 & 0 & 0 & s-1 \end{bmatrix} \overset{E}{=} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 \\ 0 & -2(s-1) & (s-1)^2 & -3(s-1) \\ 0 & 0 & 0 & s-1 \end{bmatrix}$$

$$\overset{E}{=} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 \\ 0 & 0 & (s-1)^2 & -3(s-1) \\ 0 & 0 & 0 & s-1 \end{bmatrix} \overset{E}{=} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 \\ 0 & 0 & s-1 & 0 \\ 0 & 0 & -3(s-1) & (s-1)^2 \end{bmatrix} \overset{E}{=} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & s-1 & 0 & 0 \\ 0 & 0 & s-1 & 0 \\ 0 & 0 & 0 & (s-1)^2 \end{bmatrix}$$

$$\Rightarrow J_A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

A has 3 ch. vectors corresp. to $\lambda = 1$: $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

$$\psi(s) = (s-1)^2 \Rightarrow \xi(\lambda, \mu) = \frac{\psi(\lambda) - \psi(\mu)}{\lambda - \mu} = \lambda + \mu - 2$$

$$\left.\begin{aligned} \Rightarrow \Gamma(s) &= \xi(sI, A) = (s-2)I + A \\ \Gamma'(s) &= I \end{aligned}\right\} \Rightarrow \quad \Gamma(1) = A - I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma'(1) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\Rightarrow T = \begin{bmatrix} -2 & -3 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$ where $\det(T) \neq 0$, and $AT = TJ_A$.