E6201 Linear Systems
Homework 4 (*due:* Feb. 16)

1. Suppose
$$\tilde{J} = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_p \end{bmatrix},$$
where
$$J_k = \begin{bmatrix} \lambda_k & 1 & & \\ & \lambda_k & 1 & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix}$$
denotes an $l_k \times l_k$ Jordan block, $k = 1, \ldots, p$, and $\lambda_j \neq \lambda_k,\ j \neq k$.
Show that $sI - \tilde{J}$ is equivalent to:
$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \prod_{k=1}^{p} (s - \lambda_k)^{l_k} \end{bmatrix}.$$
Hint: Matrices related by elementary operations are equivalent and matrices that have the same invariant polynomials are equivalent.

2. Let $\{t_q^{(k)}:\ q = 1, \ldots, l_k,\ k = 1, \ldots, p\}$ denote p Jordan chains, where:
$$(A - \lambda_k I)t_1^{(k)} = 0, \qquad (A - \lambda_k I)t_q^{(k)} = t_{q-1}^{(k)}, \qquad q = 2, \ldots, l_k, \qquad k = 1, \ldots, p,$$
and $\{t_1^{(k)}: k = 1, \ldots, p\}$ are linearly independent. Suppose that λ_k and l_k may be repeated. Show that $\{t_q^{(k)}: q = 1, \ldots, l_k,\ k = 1, \ldots, p\}$ are linearly independent.

3. Suppose the invariant polynomials of $sI - A$ are comprised of relatively prime factors with exponents equal to one or zero. Specifically, $h_j(s)$ is of the form:
$$h_j(s) = 1, \qquad j = 1, \ldots, k,$$
$$h_j(s) = (s - \lambda_1) \cdots \left(s - \lambda_{p_j}\right), \qquad j = k + 1, \ldots, n,$$
where $\lambda_j \neq \lambda_k,\ j \neq k$. Show that A has n-linearly independent characteristic vectors.
Hint: Show that the Jordan form of A is a diagonal matrix.

4. Find the Jordan form J_A and the similarity transformation matrix T such that $A = TJ_AT^{-1}$ for the matrix
$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$