E6201 Linear Systems
HW3 Solutions : Spring 2012

1] $B = T^{-1}AT \Rightarrow |sI-B| = |sI-T^{-1}AT| = |T^{-1}||sI-A||T|$
$\Rightarrow |sI-A| = |sI-B| \Rightarrow |sI-A| = 0$ iff $|sI-B| = 0$.

2] Let $A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & & & \\ 0 & 1 & & & \\ \vdots & & \ddots & & \\ 0 & 0 & & 1 & 0 \end{bmatrix}$.

$$\det(sI-A) = \begin{vmatrix} s+a_1 & a_2 & & a_n \\ -1 & s & & \\ & & \ddots & \\ & & -1 & s \end{vmatrix} = (s+a_1)\begin{vmatrix} s & & \\ -1 & \ddots & \\ & -1 & s \end{vmatrix} + \begin{vmatrix} a_2 & a_3 & \cdots & a_n \\ -1 & s & & \\ & & \ddots & s \end{vmatrix}$$

$$= (s+a_1)s^{n-1} + a_2\begin{vmatrix} s & & \\ -1 & \ddots & \\ & -1 & s \end{vmatrix} + \begin{vmatrix} a_3 & a_4 & & a_n \\ -1 & s & & \\ & & \ddots & s \end{vmatrix}$$

$$= s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_3\begin{vmatrix} s & & \\ -1 & \ddots & \\ & -1 & s \end{vmatrix} + \begin{vmatrix} a_4 & \cdots & a_n \\ -1 & s & \\ & \ddots & s \end{vmatrix}$$

$$= \cdots = s^n + a_1 s^{n-1} + \cdots + a_{n-2}s^2 + \begin{vmatrix} a_{n-1} & a_n \\ -1 & s \end{vmatrix}$$

$$= s^n + a_1 s^{n-1} + \cdots + a_n$$

3] $\begin{bmatrix} [C]_{i_1 j_1} & \cdots & [C]_{i_1 j_p} \\ [C]_{i_p j_1} & \cdots & [C]_{i_p j_p} \end{bmatrix} = \begin{bmatrix} \underline{\gamma}_{i_1}^T \\ \vdots \\ \underline{\gamma}_{i_p}^T \end{bmatrix} \begin{bmatrix} \underline{b}_{j_1} & \cdots & \underline{b}_{j_p} \end{bmatrix}$, where $\underline{\gamma}_i^T$ denotes the ith row of A and $\underline{b}_j$ denotes the jth col. of B

By Cauchy-Binet: $C\begin{pmatrix} i_1 \cdots i_p \\ j_1 \cdots j_p \end{pmatrix} = \sum_{1\le k_1 \le \cdots \le k_p \le n} A\begin{pmatrix} i_1 \cdots i_p \\ k_1 \cdots k_p \end{pmatrix} B\begin{pmatrix} k_1 \cdots k_p \\ j_1 \cdots j_p \end{pmatrix}$, $p \le n$.

4] If rank$(A) = r$ then A has exactly r lin. ind. cols.

$$\Rightarrow \underbrace{\exists\ A\begin{pmatrix} i_1 \cdots i_r \\ j_1 \cdots j_r \end{pmatrix} \neq 0 \text{ and } A\begin{pmatrix} i_1 \cdots i_{r+1} \\ j_1 \cdots j_{r+1} \end{pmatrix} = 0\ \forall\ i_1,\dots,i_{r+1} \in \{1,\dots,m\},\ j_1,\dots,j_{r+1} \in \{1,\dots,n\}.}_{\triangleq \text{(cond.)}}$$

Conversely, if (cond.) is true, then $\exists\ E, F$, where

$EAF = \begin{bmatrix} D & \emptyset \\ X & \emptyset \end{bmatrix}$, E is a row interchange matrix,
F is a col. combining / interchange matrix,

$D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_r \end{bmatrix}$, and X is an $(m-r)\times r$ matrix.

$\Rightarrow A$ has exactly r lin. ind. cols.