E6201 Linear Systems
Homework 3 (*due:* Feb. 9)

1. Show that similar matrices have the same characteristic values.
2. Show that every nth order monic polynomial corresponds to the characteristic polynomial on an nth order companion matrix. In particular, if $a(s) = s^n + a_1 s^{n-1} + \cdots + a_n$, then $a(s) = \det(sI - A)$, where

$$A = \begin{bmatrix} -a_1 & -a_2 & & -a_{n-1} & -a_n \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & 1 & 0 \end{bmatrix}.$$

3. Define an order-p minor of A as:

$$A\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ j_1 & j_2 & \cdots & j_p \end{pmatrix} \triangleq \begin{vmatrix} [A]_{i_1 j_1} & \cdots & [A]_{i_1 j_p} \\ \vdots & \ddots & \vdots \\ [A]_{i_p j_1} & \cdots & [A]_{i_p j_p} \end{vmatrix}.$$

Let A denote an $m \times n$ matrix and B an $n \times m$ matrix. In class we showed that if $C = AB$, then

$$\det(C) = \sum_{1 \le k_1 \le \cdots \le k_m \le n} A\begin{pmatrix} 1 & 2 & \cdots & m \\ k_1 & k_2 & \cdots & k_m \end{pmatrix} B\begin{pmatrix} k_1 & k_2 & \cdots & k_m \\ 1 & 2 & \cdots & m \end{pmatrix}, \qquad m \le n.$$

Show that an order-p minor of C can be computed as:

$$C\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ j_1 & j_2 & \cdots & j_p \end{pmatrix} = \sum_{1 \le k_1 \le \cdots \le k_p \le n} A\begin{pmatrix} i_1 & i_2 & \cdots & i_p \\ k_1 & k_2 & \cdots & k_p \end{pmatrix} B\begin{pmatrix} k_1 & k_2 & \cdots & k_p \\ j_1 & j_2 & \cdots & j_p \end{pmatrix}, \qquad p \le m \le n.$$

4. Show that the rank of A equals the largest integer r for which there exists a non-zero order-r minor of A.
5. Suppose

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_p \end{bmatrix},$$

where

$$J_k = \begin{bmatrix} \lambda_k & 1 & & \\ & \lambda_k & 1 & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix}$$ is an $l_k \times l_k$ Jordan block, $k = 1 \ldots p$, and $\lambda_j \neq \lambda_k,\ j \neq k$.

Show that $sI - J$ is equivalent to:

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \prod_{k=1}^{p}(s-\lambda_k)^{l_k} \end{bmatrix}.$$

Hint: Matrices related by elementary operations are equivalent and matrices that have the same invariant polynomials are equivalent.