E6201 Linear Systems

HW2 Solutions : Spring 2012

1] Show $\text{rank}(A) = \text{rank}(A^T)$.

Choose elementary transformation matrices $E_1, \dots, E_\ell$ and $F_1, \dots, F_m$ s.t.

$$E_\ell \cdots E_1 A F_1 \cdots F_m = \begin{bmatrix} d_1 & & & \\ & \ddots & & \\ & & d_r & \\ & & & 0 \\ & & & & 0 \end{bmatrix} \triangleq D,$$ where r denotes the rank of A

$$\Rightarrow \quad F_m^T \cdots F_1^T A^T E_1^T \cdots E_\ell^T = D$$

$$\Rightarrow \quad A^T = (F_1^T)^{-1} \cdots (F_m^T)^{-1} D (E_\ell^T)^{-1} \cdots (E_1^T)^{-1}$$
$$= (F_1^{-1})^T \cdots (F_m^{-1})^T D (E_\ell^{-1})^T \cdots (E_1^{-1})^T$$
$$= (F_m^{-1} \cdots F_1^{-1})^T D (E_1^{-1} \cdots E_\ell^{-1})^T$$

$$\Rightarrow \quad \text{rank}(A^T) = \text{rank}(D) = r$$

(since $\det((F_m^{-1} \cdots F_1^{-1})^T) = \det(F_m^{-1} \cdots F_1^{-1})$
$$= \prod_{j=1}^{m} \det(F_j^{-1}) = \prod_{j=1}^{m} \det(F_j)^{-1} \neq 0$$
and similarly $\det((E_1^{-1} \cdots E_\ell^{-1})^T) \neq 0$)

2] $$\det(sI - A) = \det\left(\left[s\underline{e}_1 - \underline{a}_1,\ s\underline{e}_2 - \underline{a}_2,\ \dots,\ s\underline{e}_n - \underline{a}_n \right]\right),$$

where $\underline{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th}$, and $\underline{a}_i$ denotes the ith column of A.

$$\Rightarrow \det(sI - A) = \det\left(\left[s\underline{e}_1, s\underline{e}_2, \dots, s\underline{e}_n \right]\right) + o(s^n)$$ ← terms of degree s^{n-1} or less

$$= s^n + o(s^n).$$

3 Let $a(s) = \det(sI - A)$, and let $v_1, \dots, v_n$ denote the n-linearly independent ch. vectors of A. Then $Av_i = \lambda_i v_i$ defines the corresponding ch. values, $\lambda_1, \dots, \lambda_n$.

$$\Rightarrow \quad a(s) = \prod_{i=1}^{n} (s - \lambda_i)$$

$$\Rightarrow \quad a(A) = \prod_{i=1}^{n} (A - \lambda_i I)$$

The factors of $a(A)$ commute, since

$$\begin{aligned}(A-\lambda_1 I)(A-\lambda_2 I) &= A^2 - \lambda_1 A - \lambda_2 A + \lambda_1\lambda_2 I \\ &= (A-\lambda_2 I)(A-\lambda_1 I)\end{aligned}$$

Let $\underline{x}$ denote an arbitrary vector in $\mathbb{R}^n$ (assumes $\{\lambda_i\}$ are real)

Then $\underline{x} = \sum_{k=1}^{n} \alpha_k \underline{v}_k$

$$\Rightarrow a(A)\underline{x} = \prod_{i=1}^{n}(A-\lambda_i I)\sum_{k=1}^{n}\alpha_k \underline{v}_k = \sum_{k=1}^{n}\alpha_k \prod_{i=1}^{n}(A-\lambda_i I)\underline{v}_k$$

$$= \sum_{k=1}^{n} \alpha_k \left[\prod_{i\neq k}(A-\lambda_i I)\right](A-\lambda_k I)\underline{v}_k = 0.$$

$$\Rightarrow a(A) = 0.$$