E6201 Linear Systems
Homework 1

1. Let $\pi(1, \dots, n)$ denote the set of permutations of the integers $1, \dots, n$. Let $\mu(p)$ denote the number of interchanges of the elements of permutation $p \in \pi(1, \dots, n)$ that result in the usual order $(1, \dots, n)$. E.g. $\mu(2,3,1,4, \dots, n) = 2$. In class we showed
$$\det(A) = \sum_{p \in \pi(1,\dots,n)} (-1)^{\mu(p)} \prod_{k=1}^{n} [A]_{k,p(k)}$$
where $[A]_{ij}$ denotes the (i, j) element of A and $p(k)$ denotes the kth element of permutation p. Show that the following equation is also true:
$$\det(A) = \sum_{p \in \pi(1,\dots,n)} (-1)^{\mu(p)} \prod_{k=1}^{n} [A]_{p(k),k}$$
[*Hint*: This can be proved from the fact that $\mu(p) = \mu(q)$, when q is the inverse permutation of p, i.e. $k = q(p(k))$, $k = 1, \dots, n$.]

2. (a) Show $\det(A^T) = \det(A)$.
(b) Show $\det(A^{-1}) = \det(A)^{-1}$ (assuming $\det(A) \neq 0$).
[*Hint*: These properties can be derived from the property $\det(AB) = \det(A)\det(B)$.]

3. A lower (upper) triangular matrix L is defined as a matrix whose elements $[L]_{ij}$ are zero for any $j > i$ $(i > j)$. Show that the determinant of an $n \times n$ lower triangular matrix is given by the product of its diagonal elements. It follows from the transpose property that the determinant of an upper triangular matrix is also given by the product of its diagonal elements. Also, the determinant of a diagonal matrix is given by the product of its diagonal elements.